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Note on the Drag experienced by a Circular Cylinder moving in a Viscous Fluid at Small Reynolds Numbers.

By

S. Tomotika and **T. Aoi**

(Received October 1, 1950)

1. Introduction

In two recent papers (**1, 2**) we have obtained the exact analytical solution of Oseen's linearized equations of motion for the case of a circular cylinder moving through an incompressible viscous fluid at small Reynolds numbers, and with the aid of this exact solution we have derived the general expression for the drag experienced by the cylinder, making special reference to the pressure drag and frictional drag separately. The values of the drag coefficient for several small Reynolds numbers have been computed numerically and it has been found that the calculated values are in good agreement with the experimental values. In calculating the values of the drag coefficient, however, a system of simultaneous linear algebraic equations has to be solved numerically so that the work is rather cumbersome. It is therefore desirable to derive, if possible, an approximate formula which may be conveniently used for computing the values of the drag, namely an expansion formula in powers of the Reynolds number*. Such an expansion formula has recently been obtained by us, starting from our general expression for the drag of the circular cylinder, though the result is unpublished yet.

Quite recently Sidrak (**4**) has made, on the basis of Oseen's linearized equations of motion as in our calculation, a similar investigation on the drag on a circular cylinder in a stream of viscous liquid at small Reynolds numbers, and has also derived an expansion formula for the drag. Although his analysis is slightly different from ours, it is naturally expected that the expansion formula derived by him should be

* As is well known, a corresponding expansion formula in powers of the Reynolds number for the drag of a sphere was obtained by Goldstein as early as 1929 (**3**).

coincident with our formula. No agreement can be found between the two formulae, however. We have therefore examined carefully Sidrak's analysis and have found out that there are unfortunately some mistakes in the course of his analysis, which give rise to an erroneous result.

The object of the present note lies in the derivation of a correct expansion formula by starting from the general expression given by Sidrak himself for the drag on a circular cylinder.

2. Sidrak's general expression for the drag on a circular cylinder

For convenience of reference we first reproduce briefly, without change in notations, the general expression for the drag on a circular cylinder which has been obtained by Sidrak by applying Filon's general analysis (5) directly to the motion of the circular cylinder.

We assume that a circular cylinder moves with constant velocity U in the positive direction of the axis of x, and let u and v denote the rectangular components of the perturbation velocity. In terms of the polar coordinates (r, θ), these components are given by

$$\left.\begin{aligned} u &= -\delta\frac{\cos\theta}{r} + \sum_{n=1}^{\infty}\frac{nb_n\cos(n+1)\theta}{r^{n+1}} \\ &\quad + e^{kr\cos\theta}\sum_{n=0}^{\infty}a_n\Big[K_n(kr)\cos n\theta + K_{n+1}(kr)\cos(n+1)\theta\Big], \\ v &= -\delta\frac{\sin\theta}{r} + \sum_{n=1}^{\infty}\frac{nb_n\sin(n+1)\theta}{r^{n+1}} \\ &\quad + e^{kr\cos\theta}\sum_{n=0}^{\infty}a_n\Big[K_{n+1}(kr)\sin(n+1)\theta - K_n(kr)\sin n\theta\Big], \end{aligned}\right\} \quad (1)$$

where $k = U/2\nu$, ν being the kinematic coefficient of viscosity of the fluid, and K_n is the modified Bessel function. The constants of integration δ, a_n and b_n are to be determined by the boundary conditions at the surface of the cylinder. If we denote the radius of the cylinder by a, the said conditions are

$$u = -U, \quad v = 0$$

at $r = a$. These conditions give the relations between the constants δ, a_n and b_n, and further it is found that the a_n's can be determined

by solving the following equations*:

$$\sum_{n=0}^{\infty} a_n \alpha_{n,m} = \begin{cases} -U & (m = 0), \\ 0 & (m = 1, 2, \cdots\cdots), \end{cases} \tag{2}$$

where

$$\alpha_{n,m} = K_n I_{n-m} + K_{n+1} I_{n+m+1},$$

the argument of both the modified Bessel functions K_n and I_n being ka.

As is well known, Filon (**5**) has discussed the forces on a cylinder of any shape and has shown that the drag is associated with a particular term in the solution of Oseen's equations, which corresponds to an inward flow along the tail and a compensating outward flow across a large contour surrounding the cylinder. In the case of the circular cylinder now under consideration, the drag on the cylinder is given by†

$$D = -4\pi\mu \sum_{n=0}^{\infty} a_n, \tag{3}$$

where μ is the coefficient of viscosity of the fluid. The constants a_n's are determined by solving the above system of simultaneous algebraic equations (2) and the general expression for the drag on the circular cylinder can thus be obtained.

3. Derivation of an expansion formula for the drag

We now proceed to the derivation of an expansion formula, starting from the general expression (3) for the drag on the cylinder. As mentioned above, the drag is given by (3) in terms of the a_n's which are to be found by solving (2). Theoretically, the solution of (2) is to be achieved by means of infinite determinants; practically, however, to find the solution numerically for any given value of ka, we solve a finite number of equations. Thus, for the first approximation, we put a_1, a_2, a_3, and so on, equal to zero, and solve the first equation for a_0. For the second approximation, we put a_2, a_3, and so on, equal to zero, and solve the first two equations for a_0 and a_1, and so on.

For our present purpose of deriving an expansion formula for the drag, we have first to express the coefficients $\alpha_{n,m}$'s in power series of

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* These equations should be compared with the similar equations (37) in our previous paper (**1**).

† It will be seen that this general expression for the drag is of a similar form to ours. See, formula (47) in our previous paper (**1**).

ka by using the expansions of the modified Bessel functions K_n and I_n*.

Writing

$$\Omega = \gamma + \log(\tfrac{1}{2}ka)$$

for the moment where $\gamma = 0.57721\cdots$ is Euler's constant, we get the following results:†

$$\begin{aligned}
\alpha_{0,0} &= K_0 I_0 + K_1 I_1 \\
&= \frac{1}{2} + \frac{3}{16}(ka)^2 + \frac{13}{384}(ka)^4 + \cdots\cdots \\
&\qquad - \Omega\left\{1 + \frac{1}{2}(ka)^2 + \frac{1}{32}(ka)^4 + \cdots\cdots\right\}, \\
\alpha_{0,1} &= K_0 I_1 + K_1 I_2 \\
&= \frac{1}{8}ka + \frac{5}{48}(ka)^3 + \cdots\cdots \\
&\qquad - \Omega\left\{\frac{1}{2}ka + \frac{1}{8}(ka)^3 + \cdots\cdots\right\}, \\
\alpha_{0,2} &= K_0 I_2 + K_1 I_3 \\
&= \frac{1}{48}(ka)^2 + \cdots - \Omega\left\{\frac{1}{8}(ka)^2 + \cdots\cdots\right\}; \\
\alpha_{1,0} &= K_1 I_1 + K_2 I_2 \\
&= \frac{3}{4} - \frac{5}{48}(ka)^2 + \cdots + \Omega\left\{\frac{1}{4}(ka)^2 + \cdots\cdots\right\}, \\
\alpha_{1,1} &= K_1 I_0 + K_2 I_3 \\
&= \frac{1}{ka} + \frac{1}{24}ka + \cdots + \Omega\left\{\frac{1}{2}ka + \cdots\cdots\right\}, \\
\alpha_{1,2} &= K_1 I_1 + K_2 I_4 \\
&= \frac{1}{2} + \cdots\cdots\cdots; \\
\alpha_{2,0} &= K_2 I_2 + K_3 I_3 \\
&= \frac{5}{12} + \cdots\cdots\cdots,
\end{aligned}$$

* Throughout our calculations the series definitions for the modified Bessel functions K_n, I_n are those adopted by G. N. Watson's '*Theory of Bessel Functions*' (Cambridge, 1922).

† The corresponding expressions are not given in Sidrak's paper (4).

$$\alpha_{2,1} = K_2 I_1 + K_3 I_4$$

$$= \frac{1}{ka} + \cdots\cdots\cdots,$$

$$\alpha_{2,2} = K_2 I_0 + K_3 I_5$$

$$= \frac{2}{(ka)^2} + \cdots\cdots\cdots.$$

Assuming ka to be small we can easily see that the orders of a_0, a_1, a_2, $\cdots$ are 1, $(ka)^2$, $(ka)^4$, $\cdots$ respectively. Therefore our method of approximation is justified.

As usual we introduce here the non-dimensional drag coefficient C_D defined by

$$C_D = \frac{D}{\rho U^2 d}, \tag{4}$$

where ρ is the density of the fluid and d the diameter of the cylinder so that $d = 2a$. Then, the general expression for C_D becomes

$$C_D = -\frac{4\pi}{UR} \sum_{n=0}^{\infty} a_n, \tag{5}$$

where $R = 4ka = Ud/\nu$ is the Reynolds number.

We have obtained the first, second and third approximate expressions for C_D respectively. The first approximation gives

$$C_D = \frac{4\pi}{RA}, \tag{6}$$

where we have written A for $\frac{1}{2} - \Omega$ so that

$$A = \frac{1}{2} - \gamma - \log\frac{R}{8} = 2.0022 - \log R. \tag{7}$$

Formula (6) is nothing but the well-known result obtained by Lamb.

Next, the second approximation yields an expansion formula of the form:

$$C_D = \frac{4\pi}{RA}\left[1 - \frac{1}{A}\left(A^2 - \frac{1}{2}A + \frac{5}{16}\right)\frac{R^2}{32}\right]. \tag{8}$$

Further, the third approximation gives*

* Sidrak has gone as far as the fourth approximation, but it will be seen in the subsequent section that the third approximation is quite sufficient for computing the values of C_D provided that the Reynolds number R is less than 4.

$$C_D = \frac{4\pi}{RA}\Big[1 - \frac{1}{A}\Big(A^2 - \frac{1}{2}A + \frac{5}{16}\Big)\frac{R^2}{32} - \frac{1}{A^2}\Big(A^4 - \frac{1}{3}A^3 + \frac{7}{72}A - \frac{25}{256}\Big)\frac{R^4}{32^2}\Big]. \qquad (9)$$

It is found that this formula is in accordance with that obtained by us previously from our general expression (47) for the drag in our former paper (**1**).*

4. Numerical computations

It will be important to estimate the range of values of the Reynolds number in which our expansion formula (9) just obtained is applicable. Such an estimation can be made by examining the convergency of the expansion formula and also by comparing the values of the drag coefficient calculated by the expansion formula (9) with those calculated numerically with the aid of the general expression (5).

As for the convergency of our formula, it is first seen that in the neighbourhood of $R = 7.4$ at which A becomes zero, our formula is evidently divergent. In Table I the numerical values of the three terms in the square brackets in the formula (9) are given for various values of the Reynolds number R less than 7. The third column gives the values of the second term:

$$-\frac{1}{A}\Big(A^2 - \frac{1}{2}A + \frac{5}{16}\Big)\frac{R^2}{32},$$

while the fourth column gives the values of the third term:

$$-\frac{1}{A^2}\Big(A^4 - \frac{1}{3}A^3 + \frac{7}{72}A - \frac{25}{256}\Big)\frac{R^4}{32^2}.$$

* The corresponding expression obtained by Sidrak will be given here, for comparison, though it is not correct. Thus,

$$C_D = \frac{4\pi}{RA}\Big[1 - \Big(\frac{R}{8}\Big)^2\Big(2A + \frac{9}{8A} - \frac{\gamma}{A} - 1\Big) - \Big(\frac{R}{8}\Big)^4\Big(4A^2 + 4\gamma A - \frac{9}{32A^2} + \frac{9\gamma}{4A^2} - \frac{\gamma^2}{A^2} - \gamma - \frac{\gamma}{A} + \frac{8}{9A} - \frac{10}{3A} + \frac{1}{2}\Big) - \cdots\cdots\Big],$$

$\gamma = 0.57721\cdots$ being as before Euler's constant.

It will be seen that our formula (9) begins to diverge at $R = 4$, below which, however, our formula converges rather rapidly.

TABLE I

R	1st term	2nd term	3rd term
0.4	1	− 0.01	− 0.00
0.6	1	− 0.02	− 0.00
0.8	1	− 0.04	− 0.00
1	1	− 0.05	− 0.00
2	1	− 0.13	− 0.02
3	1	− 0.21	− 0.04
4	1	− 0.31	− 0.02
5	1	− 0.54	0.22
6	1	− 1.34	2.24
7	1	− 7.82	68.21

The values of the drag coefficient calculated respectively by the first, second and third approximate formulae are given in Table II and are shown graphically in Fig. 1. In the last column of this table are also shown the values of the drag coefficient calculated numerically by the general formula (5).* It will be seen that for the values of the Reynolds number less than 4, our third approximate expansion formula (9) gives, with sufficient accuracy, the values of the drag coefficient of the circular cylinder.†

TABLE II Values of C_D

R	1st approx.	2nd approx.	3rd approx.	Calculated by (5)
0.4	10.76	10.63	10.63	10.63
0.6	8.33	8.13	8.13	8.13
0.8	7.06	6.80	6.78	6.78
1	6.28	5.95	5.93	5.93
2	4.80	4.17	4.07	4.04
3	4.64	3.66	3.47	3.39
4	—	3.51	3.42	2.92

* The values of the drag coefficient calculated numerically by the general formula (5) are exactly the same as those calculated by formula (48) in our previous paper (**1**). See Table I there.

† It is merely accidental that the values of the drag coefficient computed by Sidrak by his own expansion formula differ only slightly from those calculated by our third approximate formula. See Table I in his paper (**4**).

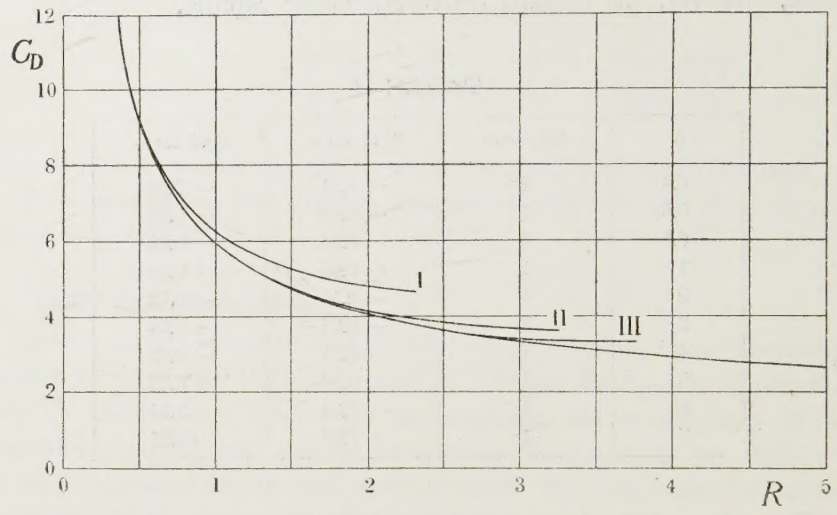

Fig. 1.

5. Summary and conclusion

In the present note an expansion formula correct to the fourth power of the Reynolds number R is derived by starting from the general expression for the drag on a circular cylinder obtained recently by Sidrak, and it is found that the formula thus obtained agrees with that obtained by us previously, which is not yet published.

Sidrak himself has also derived a similar expansion formula for the drag coefficient by making use of his general formula. Notwithstanding that our analysis is developed along the same lines as Sidrak's, no agreement is found between our expansion formula and the corresponding Sidrak's formula. By careful examination of Sidrak's analysis, it has been found that his result is unfortunately erroneous. It seems that the source of error lies in his use of incorrect expansions for the modified Bessel function K_n.

Some numerical discussions are made on the values of the drag coefficient by using our expansion formula, and the range of values of the Reynolds number R is estimated in which our expansion formula is applicable. It is thus found that our expansion formula correct to the order of R^4 can be used with sufficient accuracy, provided that the Reynolds number assumes the values less than 4.

6. Acknowledgements

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On the Two-dimensional Supersonic Free Gas Jet.*

By

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1. Introduction and summary

Theoretical studies on the two-dimensional supersonic free jet of a perfect gas were made by Prandtl (1) in 1904, and the ratio of the breadth of jet to the wave-length of the wavy motion has been obtained as a function of the Mach number associated with the mean-stream velocity. In his paper, however, the deformations of the free-streamline boundaries from the straight lines were neglected. Later, Riabouchinsky (2) treated the same problem by a different method under a similar assumption that the flow is almost uniform, and obtained the streamlines and the isobars. The approximation of his analysis is considered to be better than that of Prandtl. However, a hypothetical gas employed by him is essentially of the same approximate nature as Kármán-Tsien's hypothetical gas (3) whose pressure-density curve has a common tangent but different curvature with the adiabatic pressure-density curve.

In the present paper, a certain appropriate equation of state-change of gas is assumed so that the fundamental equations for the stream function and the velocity potential might easily be solved exactly. A hypothetical gas such introduced can approximate in fact the real gas obeying the adiabatic law better than Kármán-Tsien's hypothetical gas, since its pressure-density curve can be made to coincide up to its curvature with the corresponding curve for the real gas. Of course, the validity of the present theory is limited to some range of speeds over which both our hypothetical gas and the real gas have similar properties. But, in this range, once the fundamental equations of motion for our hypothetical gas be solved exactly, it will be possible to infer, from the results obtained, more reliably the general behaviours of the corresponding real gas flow, rather than from the approximate solutions

* Communicated by Professor S. Tomotika.

of the exact equations of motion of the real gas which obeys the adiabatic law.

It is shown that the fundamental equation for the stream function governing the flow of our hypothetical gas has a family of exact solutions of simple form, and combing some of appropriate solutions we can construct the solution for the supersonic free jet. In this case, we meet with a problem how to join two hodograph solutions on a certain curve. This is done, in our problem, on a characteristics of the fundamental hodograph equations. The special characteristics on which the hodograph solution branches are called the branch-line. When two solutions are joined on the branch-line, there occur in general discontinuities of the curvature of the streamline and of the velocity gradient in the physical plane. In the present paper, two solutions are joined on the branch-line in such a way that at least the curvature of the streamline and the velocity gradient are continuous there. It will be one of the important future problems to discuss how we must join two hodograph solutions so that they represent a perfectly regular flow in the physical plane.

A detailed numerical disccussion has been made in a special case, giving the Mach lines networks, the isobars and the curves of the constant direction of velocity vectors in the physical plane. In the appendix, a general relationship between the breadth, the wave-length, the flow quantity of jet and the Mach number of the mean-flow is given for a group of solutions representing the supersonic free jet. This relation is considered to be an extension of Prandtl's formula and, when the deformation of the free-streamline boundaries is neglected, it degenerates into his formula.

The writer wishes to express his cordial thanks to Professor Dr. S. Tomotika for his continued interest throughout this work, and also to Dr. K. Tamada for his kind suggestions.

2. Fundamental equations

It is well known that when the magnitude, q, and the angle of inclination, θ, of the velocity vector are taken as independent variables, the equations of steady irrotational motion of a perfect gas in two-dimensions are given by

$$\Phi_q = q\frac{d}{dq}\left(\frac{\rho_0}{\rho q}\right)\Psi_\theta\,, \quad \Phi_\theta = \frac{\rho_0 q}{\rho}\Psi_q\,, \tag{2.1}$$

where Φ and Ψ are the velocity potential and the stream function for the flow respectively, and ρ and ρ_0 are the density at any point and at the stagnation point in the flow field respectively. Suffixes q, θ in Φ_q, Ψ_θ, Φ_θ, Ψ_q denote partial differentiation with respect to q or θ.

Instead of q, we introduce a new variable τ defined by

$$\tau = -\int^q (-K)^{\frac{1}{2}} \frac{\rho}{\rho_0 q}\, dq\,, \tag{2.2}$$

with

$$K = -\frac{\rho_0 q^2}{\rho}\frac{d}{dq}\left(\frac{\rho_0}{\rho q}\right). \tag{2.3}$$

Now, let us assume that the pressure, p, of the gas is a function of the density ρ only. Then, by making use of Bernoulli's equation in differential form: $qdq + dp/\rho = 0$, we get $d\rho/dq = -\rho q/c^2$, where $c = \sqrt{dp/d\rho}$ is the local speed of sound. Thus, K is written as

$$K = \left(1 - \frac{q^2}{c^2}\right)\left(\frac{\rho_0}{\rho}\right)^2. \tag{2.4}$$

If a certain equation of state of the gas is assumed, K becomes a definite function of q (and therefore, of τ) only.

Equations (2.1) are now transformed into

$$\left.\begin{aligned} \Phi_\tau &= -\chi^2 \Psi_\theta\,, \quad \Phi_\theta = -\chi^2 \Psi_\tau\,, \\ \chi &= (-K)^{\frac{1}{2}}\,. \end{aligned}\right\} \tag{2.5}$$

Further, in place of Φ and Ψ, we use, with Professor I. Imai (4), ϕ and ψ as dependent variables which are connected with Φ and Ψ as:

$$\Phi = \chi\phi\,, \qquad \Psi = \chi^{-1}\psi\,. \tag{2.6}$$

Then equations (2.5) become

$$\phi_{\tau\tau} - \phi_{\theta\theta} = \chi\frac{d^2\chi^{-1}}{d\tau^2}\phi\,, \tag{2.7}$$

$$\psi_{\tau\tau} - \psi_{\theta\theta} = \chi^{-1}\frac{d^2\chi}{d\tau^2}\psi\,. \tag{2.8}$$

As far as we confine ourselves to the adiabatic gas flow, no simple exact solutions can be obtained in general. Therefore, we now introduce such a hypothetical gas that enables the above equations to be solved simply. Thus, we assume that

$$\chi = c_1\tau + c_2 \tag{2.9}$$

where c_1, c_2 are, for the present, arbitrary constants. Then, the equation for ψ takes the form:

$$\psi_{\tau\tau} - \psi_{\theta\theta} = 0\,,$$

and the general solution of this equation is given by

$$\psi = f(\tau + \theta) + g(\tau - \theta)$$

where f, g are both arbitrary functions. Thus the expression for the function Ψ becomes

$$\Psi = \chi^{-1}\{f(\tau + \theta) + g(\tau - \theta)\}\,, \tag{2.10}$$

while the corresponding expression for Φ takes the form:

$$\Phi = \chi\{-f(\tau + \theta) + g(\tau - \theta)\} + c_1\int^{\theta}\{f(\tau + \theta) + g(\tau - \theta)\}d\theta\,. \tag{2.11}$$

3. Equations of state of our hypothetical gas

The equation of state-change of our hypothetical gas can be obtained by a method used by Imai (5).

Now we put $\Psi = Q(q)\cos\theta$, where $Q(q)$ is a function of q only. Then, the equation for determining Q becomes linear and has two independent solutions: $1/q$ and p/q, where p is the pressure. Hence, if we write $\Psi = T(\tau)\cos\theta$, each of the two functions $1/q$ and p/q must be expressed as a linear combination of the solutions of the equation for determining $T(\tau)$, namely:

$$\frac{d^2T}{d\tau^2} + \frac{2c_1}{c_1\tau + c_2}\frac{dT}{d\tau} + T = 0\,. \tag{3.1}$$

For convenience's sake, we use, instead of τ, another new variable t defined as:

$$t = \tau + \frac{c_2}{c_1}\,. \tag{3.2}$$

Then, the general solution of equation (3.1) is given by $At^{-1}\sin(t + \varepsilon)$, where A and ε are arbitrary constants of integration. Hence, we get

$$\frac{1}{q} = A\frac{\sin(t + \varepsilon)}{t}\,, \qquad \frac{p}{q} = B\frac{\sin(t + \delta)}{t}\,, \tag{3.3}$$

B and δ being also constants.

Assuming that $p = p_1$, $t = t_1$, $\tau = \tau_1$ at $q = q_1$, we get

$$\frac{q}{q_1} = \frac{t \sin(t_1 + \varepsilon)}{t_1 \sin(t + \varepsilon)}, \qquad \frac{p}{p_1} = \frac{\sin(t_1 + \varepsilon)\sin(t + \delta)}{\sin(t + \varepsilon)\sin(t_1 + \delta)}, \tag{3.4}$$

with

$$t_1 = \tau_1 + \frac{c_2}{c_1}. \tag{3.5}$$

The density ρ can be obtained from (2.2) as follows:

$$\frac{\rho}{\rho_1} = \frac{t_1 \sin(t + \varepsilon)\sin(t_1 + \varepsilon) - t_1 \cos(t_1 + \varepsilon)}{t \sin(t_1 + \varepsilon)\sin(t + \varepsilon) - t \cos(t + \varepsilon)}, \tag{3.6}$$

where we have assumed that $\rho = \rho_1$ at $q = q_1$.

Finally, the local Mach number M, defined as q/c, is given by

$$M^2 = 1 + \left\{\frac{t}{1 - t\cot(t + \varepsilon)}\right\}^2. \tag{3.7}$$

4. Determination of constants τ_1, c_1, c_2, ε, δ

For our hypothetical gas, χ and τ are related by (2.9), while for the real gas obeying the adiabatic law, the corresponding quantities, denoted by χ_a and τ_a respectively, are related indirectly by the two equations:

$$\left.\begin{aligned} &\chi_a = (\alpha^2 - 1)^{-\frac{1}{4}(\alpha^2-1)} \zeta^{\frac{1}{2}} (\alpha^2 + \zeta^2)^{\frac{1}{4}(\alpha^2-1)}, \\ &\tau_a = \tan^{-1}\zeta - \alpha \tan^{-1}\frac{\zeta}{\alpha}, \\ &\text{where} \quad \zeta = \left(\frac{\alpha^2 z - 1}{1 - z}\right)^{\frac{1}{2}}, \quad z = \frac{q^2}{q_{\max}^2}, \quad \alpha^2 = \frac{\gamma + 1}{\gamma - 1}, \end{aligned}\right\} \tag{4.1}$$

$q_{\max}$ being the theoretically attainable maximum value of q, and γ the ratio of the specific heats of the real gas subject to the adiabatic law.

Now, we choose the curve of the state-change for our hypothetical gas as follows. First, we assume that τ is equal to τ_a at $q = q_1$, namely:

$$\tau_1 = \tan^{-1}\zeta_1 - \alpha \tan^{-1}\frac{\zeta_1}{\alpha}, \quad \zeta_1 = \left(\frac{\alpha^2 z_1 - 1}{1 - z_1}\right)^{\frac{1}{2}}, \quad z_1 = \frac{q_1^2}{q_{\max}^2}. \tag{4.2}$$

Further, we assume that at the point $\tau = \tau_1$ the χ-τ curve coincides with the corresponding χ_a-τ_a curve for the real gas up to their tangents; i. e., we assume that

$$(\chi_a)_{\tau_a=\tau_1} = c_1\tau_1 + c_2\,, \qquad \left(\frac{d\chi_a}{d\tau_a}\right)_{\tau_a=\tau_1} = c_1\,.$$

Then we get

$$\left.\begin{aligned} c_1 &= -\frac{1}{2}\alpha^2(\alpha^2-1)^{-\frac{1}{4}(\alpha^2+3)}\zeta_1^{-\frac{5}{2}}(\alpha^2+\zeta_1^2)^{\frac{1}{4}(\alpha^2-1)}(1+\zeta_1^2)^2, \\ c_2 &= (\alpha^2-1)^{-\frac{1}{4}(\alpha^2-1)}\zeta_1^{\frac{1}{2}}(\alpha^2+\zeta_1^2)^{\frac{1}{4}(\alpha^2-1)} - c_1\tau_1\,, \end{aligned}\right\} \quad (4.3)$$

and t_1 can be determined by (3. 5).

Next, we proceed to the determination of ε and δ. Since p, q and ρ must obey Bernoulli's equation in differential form: $q\,dq + dp/\rho = 0$, if we substitute (3. 3), (3. 4) and (3. 6) in this equation and put p_1 and ρ_1 equal to the corresponding quantities for the real gas respectively, we get following relation:

$$\sin(\varepsilon-\delta) + \frac{2\gamma}{\gamma-1}\frac{z_1}{1-z_1}\frac{\sin(t_1+\delta)}{t_1} \times \left\{\sin(t_1+\varepsilon) - t_1\cos(t_1+\varepsilon)\right\} = 0\,. \quad (4.4)$$

Further, let the tangent to the ρ-q curve for our hypothetical gas at the point $q = q_1$, $\rho = \rho_1$, be coincident with the tangent to the ρ-q curve for the real gas. Then it follows that

$$\left\{\frac{t_1}{1-t_1\cot(t_1+\varepsilon)}\right\}^2 = \frac{2}{\gamma-1}\frac{z_1}{1-z_1} - 1\,. \quad (4.5)$$

It will be seen that this condition is equivalent to assuming that for both our hypothetical gas and the real gas the respective Mach numbers at $q = q_1$ coincide with each other.

The five equations (4. 2)—(4. 5) enable us to determine five constants τ_1, c_1, c_2, ε, δ involved in the equations of state of our hypothetical gas. Since we have assumed that at $q = q_1$ the χ-τ curve coincides with the χ_a-τ_a curve up to their tangents, it is readily proved that the p-ρ curves for both gases coincide with each other up to their curvatures. Therefore, our present hypothetical gas is a better approximation to the real gas near the point $q = q_1$ than Kármán-Tsien's hypothetical gas. Fig. 1 shows the curve of p/p_1 plotted against ρ_1/ρ in a special

case when $M_1 = \sqrt{5}$. It will be seen that our hypothetical gas approximates excellently the real gas.

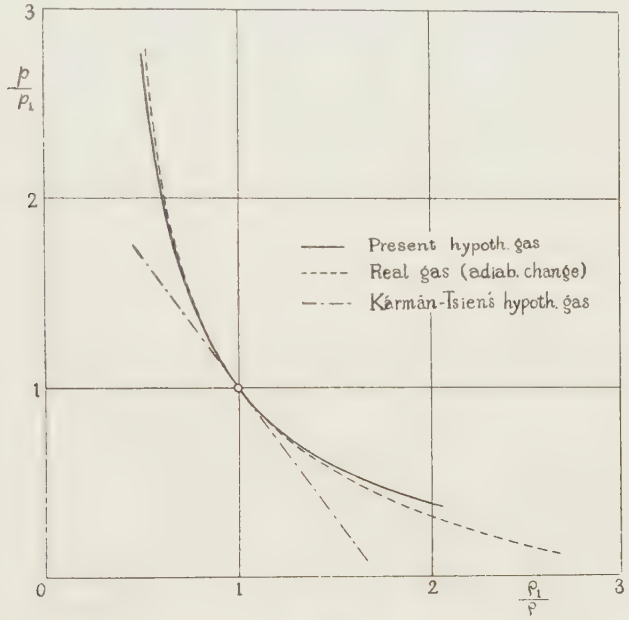

Fig. 1.

5. Solution for the supersonic free gas jet

The streamlines in the two-dimensional wavy flow of supersonic free jet are expressed, in the hodograph plane, as a family of closed curves round a point which corresponds to the mean velocity. It is well known that if a streamline touches one of the characteristics of the fundamental equation (2. 8), there appears in general a turn back of the streamline in the physical plane and a limit line is formed. It has been found, however, that when the envelope of the family of the streamlines in the hodograph plane coincides with one of the characteristics, the limit line does not appear in the physical plane. Therefore, we now search for a family of closed streamlines in the hodograph plane which satisfy the above condition.

For convenience, we introduce new variables ξ, η in place of τ, θ as follows:

$$\xi = \frac{1}{2\sigma}\left\{\tau_1 + \sigma - (\tau + \theta)\right\}, \quad \eta = \frac{1}{2\sigma}\left\{\tau_1 + \sigma - (\tau - \theta)\right\}. \quad (5.1)$$

As will be seen presently, σ is a constant which is connected with the deviation of the velocity vector from that of the mean-flow. The characteristics of the equation (2. 8) are expressed by the equations $\tau \pm \theta =$ const and accordingly also by $\xi =$ const and $\eta =$ const. Now, we consider in the hodograph plane a square region with centre at the point $\tau = \tau_1$, $\theta = 0$ whose sides are given by $\tau + \theta = \tau_1 \pm \sigma$ and $\tau - \theta = \tau_1 \pm \sigma$, namely by $\xi = 0, 1$ and $\eta = 0, 1$ (Fig. 2). We assume that all the streamlines in the hodograph plane are simply closed curves touching the sides of the square as shown in Fig. 3. Also we suppose

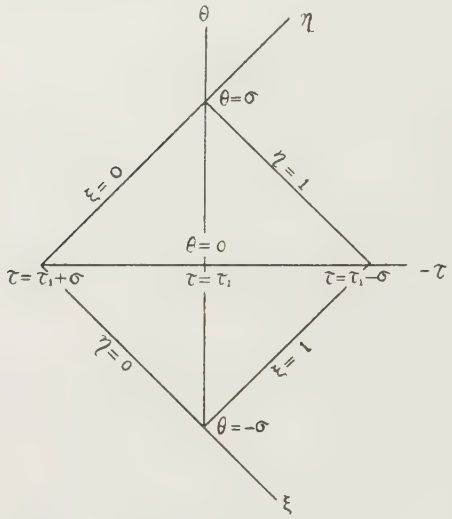

Fig. 2.

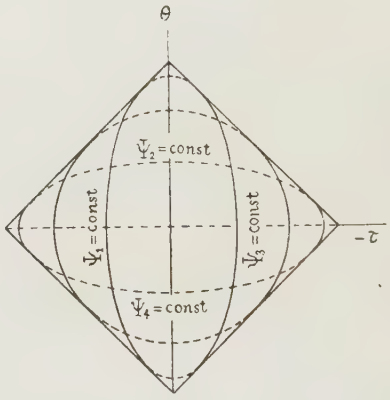

Fig. 3.

that these closed streamlines are transformed into the streamlines in a wavy flow of jet in the physical plane. Figs. 4 (a) and 4 (b) represent the correspondence between two planes. The origin of the physical plane is taken at the mid-point of the narrowest portion of the jet, and the x- and y-axes are taken parallel and perpendicular to the main-stream respectively. The line ABCD denotes the free-streamline boundary on which $q = q_1$, and it corresponds to the segment ABCD in the hodograph plane on which $\tau = \tau_1$. If we denote the points of the minimum and maximum velocities on the x-axis by O and O′ respectively, these correspond to the points $\tau = \tau_1 \pm \sigma$, $\theta = 0$ in the hodograph plane. The lines OB and BO′ in the xy-plane are the particular Mach lines passing through O and O′, and they correspond to the sides OB, BO′ of the square region in the $\tau\theta$-plane. The regions denoted by 1, 2, 3, 4 in the xy-plane correspond respectively to the triangular regions:

OBD ($\xi + \eta \leqq 1$, $\xi \geqq 0$, $\eta \geqq 0$), OBO′ ($\xi \leqq \eta$, $\xi \geqq 0$, $\eta \leqq 1$), BO′D ($\xi + \eta \geqq 1$, $\xi \leqq 1$, $\eta \leqq 1$) and O′DO ($\xi \geqq \eta$, $\xi \leqq 1$, $\eta \geqq 0$) in the hodograph plane. We denote the velocity potentials and the stream functions in these regions by Φ_1, Ψ_1; Φ_2, Ψ_2; Φ_3, Ψ_3; Φ_4, Ψ_4 respectively.

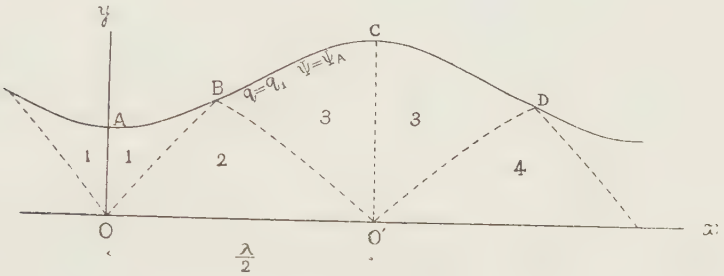

Fig. 4. (a)

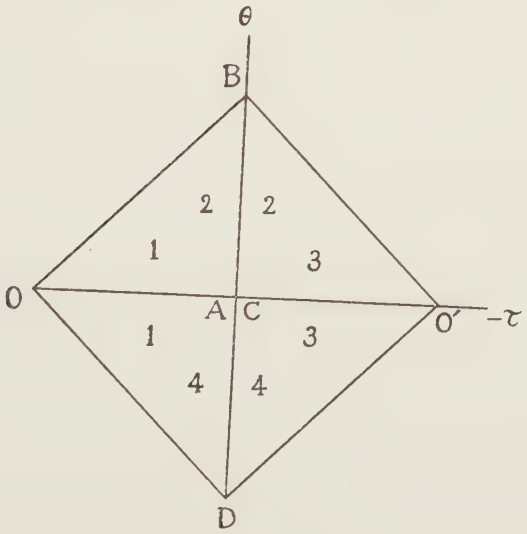

Fig. 4. (b)

To determine $\Psi_i (i = 1, 2, 3, 4)$ we shall take account of the physical conditions that the flow in the physical plane is symmetrical with respect to both the x-, y-axes and that the flow is bounded by two free-stream-lines. These conditions, together with the continuity of the stream-lines, are stated as follows:

$$\left.\begin{aligned}
&(\mathrm{i})\ \ \Psi_1(\xi,\eta)=\Psi_1(\eta,\xi),\ \ \Psi_1(\xi,1-\xi)=\text{const},\ \ \Psi_1(0,0)=0\,;\\
&(\mathrm{ii})\ \ \Psi_2(0,\eta)=\Psi_1(0,\eta),\ \ \Psi_2(\xi,\xi)=0\,;\\
&(\mathrm{iii})\ \ \Psi_3(\xi,1)=\Psi_2(\xi,1),\ \ \Psi_3(\xi,1-\xi)=\Psi_1(\xi,1-\xi)\,;\\
&(\mathrm{iv})\ \ \Psi_4(1,\eta)=\Psi_3(1,\eta),\ \ \Psi_4(\xi,\xi)=0\,.
\end{aligned}\right\}\tag{5.2}$$

These conditions determine Ψ_i's as follows:

$$\left.\begin{aligned}
&\Psi_1=\chi^{-1}\{F(\xi)+F(\eta)\}\,, && \Psi_2=\chi^{-1}\{-F(\xi)+F(\eta)\}\,,\\
&\Psi_3=\chi^{-1}\{-F(\xi)-F(\eta)+2k\}\,, && \Psi_4=\chi^{-1}\{F(\xi)-F(\eta)\}\,,
\end{aligned}\right\}\tag{5.3}$$

where

$$F(+0)=0\,,\quad F(\xi)+F(1-\xi)=\text{const}=k\quad(0\leqq\xi\leqq1)\,.\tag{5.4}$$

Also, Φ_i's are obtained in the forms:

$$\left.\begin{aligned}
\Phi_1&=\chi\{-F(\xi)+F(\eta)\}-2c_1\sigma\int_0^{\xi}F(\xi)d\xi+2c_1\sigma\int_0^{\eta}F(\eta)d\eta\,,\\
\Phi_2&=\chi\{F(\xi)+F(\eta)\}+2c_1\sigma\int_0^{\xi}F(\xi)d\xi+2c_1\sigma\int_0^{\eta}F(\eta)d\eta\,,\\
\Phi_3&=\chi\{F(\xi)-F(\eta)\}+2c_1\sigma\int_0^{\xi}F(\xi)d\xi-2c_1\sigma\int_0^{\eta}F(\eta)d\eta\\
&\qquad-2c_1\sigma k(\xi-\eta)+2k(c_1\tau_1+c_2)\,,\\
\Phi_4&=\chi\{-F(\xi)-F(\eta)\}-2c_1\sigma\int_0^{\xi}F(\xi)d\xi-2c_1\sigma\int_0^{\eta}F(\eta)d\eta\\
&\qquad+c_1\sigma k+4k(c_1\tau_1+c_2)\,.
\end{aligned}\right\}\tag{5.5}$$

6. Conditions for F

As for the conditions on the branch-lines which are the Mach lines passing through O and O′, we have only assumed that the stream function and the velocity potential are continuous. Therefore, although the velocity vector and accordingly the tangent to the stream-line are both continuous near the branch-line, it is not yet certain whether the velocity gradient and the curvature of the streamline are continuous or not, and whether the turn back of the streamline occurs or not.

If we denote the line element of a streamline by ds, the velocity gradient along the streamline and the curvature of the streamline are expressed respectively by

$$\frac{dq}{ds} = \frac{1}{2}\sigma q \frac{dq}{d\tau}\frac{1}{\chi^2}\frac{\Psi_\eta - \Psi_\xi}{\Psi_\xi \Psi_\eta},$$

$$\frac{d\theta}{ds} = \frac{1}{2}\sigma q \frac{1}{\chi^2}\frac{\Psi_\xi + \Psi_\eta}{\Psi_\xi \Psi_\eta}.$$

Substituting (5. 3) in these equations, we get, on the branch-line $\xi = 0$,

$$\left(\frac{dq}{ds}\right)_{\xi=+0} = \frac{1}{2}\sigma\left(q\frac{dq}{d\tau}\right)_{\xi=0} \frac{(\chi)_{\xi=0}[F''(\eta) \mp F''(+0)]}{[c_1\sigma F(\eta) + (\chi)_{\xi=0}F''(\eta)][c_1\sigma F(\eta) \pm (\chi)_{\xi=0}F''(+0)]},$$

$$\left(\frac{d\theta}{ds}\right)_{\xi=+0} = \frac{1}{2}\sigma(q)_{\xi=0} \frac{2c_1\sigma F(\eta) + (\chi)_{\xi=0}[F''(\eta) \pm F''(+0)]}{[c_1\sigma F(\eta) + (\chi)_{\xi=0}F''(\eta)][c_1\sigma F(\eta) \pm (\chi)_{\xi=0}F''(+0)]},$$

where the upper of the double signs denotes the limit from the region 1, while the lower the limit from the region 2. Accordingly, in order that the velocity gradient and the curvature of the streamline may be continuous, it must be that

$$F''(+0) = 0 \qquad \text{or} \qquad F''(+0) = \infty.$$

The same conditions are also obtained when we consider the other branch-lines: $\eta = 1$, $\xi = 1$ and $\eta = 0$.

Next, along a streamline we get

$$\left(\frac{dx}{d\xi}\right)_{\xi=+0} = -2\left(\frac{\cos\theta}{q}\right)_{\xi=0}[c_1\sigma F(\eta) \pm (\chi)_{\xi=0}F''(+0)],$$

the double signs being of the same meaning as before. Thus, when $F''(+0) = 0$, $dx/d\xi$ is finite and of equal amount when approached to $\xi = 0$ from either region, and therefore the streamline turns back in the physical plane. On the other hand, when $F''(+0) = +\infty$, we have $(dx/d\xi)_{\xi\to+0} \to \mp\infty$, and therefore the streamline does not turn back and a smooth flow is obtained. Hence, we impose upon the function F the following conditions:

$$F(+0) = 0,\ F''(+0) = +\infty,\ F(\xi) + F(1-\xi) = k\ (0 \leqq \xi \leqq 1). \quad (6.1)$$

If we adopt an appropriate one-valued, continuous and smooth function $F(\xi)$ satisfying these conditions, we get, by starting with the stream functions Ψ_i's given in (5. 3), a supersonic free jet flow which is continuous up to the velocity gradient and the curvature of the streamline.

7. An example

As an example, we assume that

$$F(\xi) = \sqrt{\xi} - \sqrt{1-\xi} + 1\,, \tag{7.1}$$

which satisfies the conditions (6. 1). Substituting this into (5. 3) and (5. 5), we obtain the expressions for Ψ_i's and Φ_i's. Transformation into the physical plane is made by integrating the following equations:

$$dx = \frac{1}{q}\cos\theta\, d\Phi - \frac{\rho_0}{\rho q}\sin\theta\, d\Psi\,,$$

$$dy = \frac{1}{q}\sin\theta\, d\Phi + \frac{\rho_0}{\rho q}\cos\theta\, d\Psi\,.$$

It is found that the flow thus obtained expresses in fact a supersonic free jet changing its state of motion periodically in the direction of the x-axis. We denote the wave length of the jet, its mean breadth, the deviation of the breadth from its mean value by λ, b and Δ respectively. Then, they are defined by

$$\lambda = 2x_{O'}\,, \qquad b = y_A + y_C\,, \qquad \Delta = y_C - y_A\,, \tag{7.2}$$

where suffixes O′, A, C denote the points at which the values of the coordinates are to be taken (Fig. 4 (a)). For the flow deduced from (7. 1), it follows that when σ is small,

$$\left.\begin{aligned} \frac{2b}{\lambda} &= \frac{1}{\sqrt{M_1^2-1}}\left\{1 + \frac{7(\gamma+1)M_1^4}{120(M_1^2-1)}\sigma^2 + O(\sigma^4)\right\}, \\ \frac{2\Delta}{\lambda} &= \frac{1}{3}(2\sqrt{2}-1)\sigma + \frac{1}{315}(25\sqrt{2}-11)\sigma^3 + O(\sigma^5)\,, \end{aligned}\right\} \tag{7.3}$$

M_1 being the Mach number associated with the free-streamline on which $q = q_1$.

Numerical calculations have been worked out for a particular case when

$$M_1 = \sqrt{5} \qquad \text{with} \qquad \gamma = 1.4\,,$$

i. e., when

$$\zeta_1 = 2\,, \qquad z_1 = \frac{1}{2} \qquad \text{and} \qquad \alpha = \sqrt{6}\,.$$

In this case, constants τ_1 etc. are obtained as follows:

$$\tau_1 = -0.570077\,, \qquad t_1 = -0.533333\,,$$
$$c_1 = -6.306724\,, \qquad c_2 = -0.231732\,,$$
$$\varepsilon = -0.095463\,, \qquad \delta = 0.973171\,.$$

The constant σ, which is a measure of the deviation of the velocity from the mean velocity, has been chosen as

$$\sigma = 0.2\,,$$

when the speed of flow varies in the range $0.884 < q/q_1 < 1.097$.

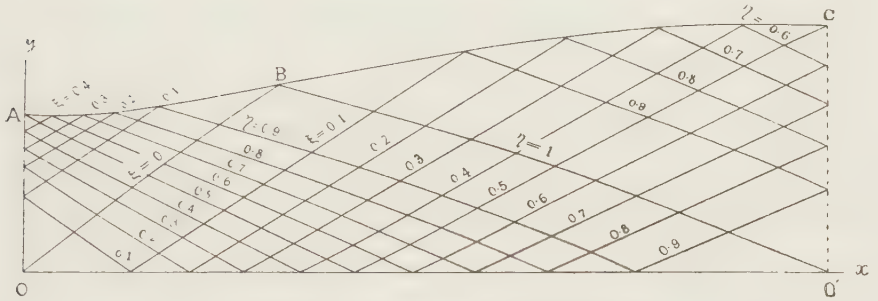

Fig. 5. Two families of Mach lines.

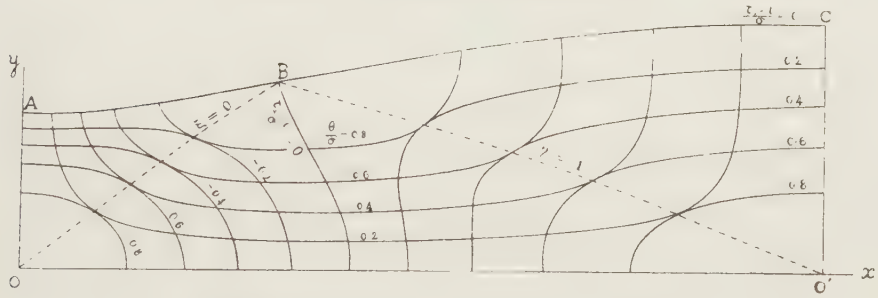

Fig. 6. Curves of $\tau = \text{const}$ and $\theta = \text{const}$.

Fig. 5 shows the two families of the Mach lines. The Mach lines marked with $\xi = 0$ and $\eta = 1$ represent the branch-lines. Fig. 6 shows the curves of the constant speed ($\tau = \text{const}$) and those of the constant direction ($\theta = \text{const}$). These two families contact with each other on the branch-lines.

8. Appendix

Starting with the solution obtained in § 5, we get the relations between λ, b and Δ as follows:

$$\left.\begin{aligned}\frac{2b}{\lambda} &= \frac{1}{\sqrt{M_1^2-1}} + \frac{(\gamma+1)M_1^4}{(M_1^2-1)^{\frac{3}{2}}}\frac{\sigma S}{k-4\sigma S'}\\ \frac{2\Delta}{\lambda} &= \frac{4\sigma C}{k-4\sigma S'}\end{aligned}\right\} \quad (8.1)$$

where

$$S=\int_0^{\frac{1}{2}} F(\xi)\sin\sigma(1-2\xi)d\xi\,,\quad C=\int_0^{\frac{1}{2}} F(\xi)\cos\sigma(1-2\xi)d\xi\,.$$

Also, we denote by Q the quantity of mass carried by the jet in unit time, which may be defined by $Q=2\rho_0\Psi_A$ with the value Ψ_A of the stream function on the free streamline ABCD. Then we obtain, after some reductions,

$$Q=bq_1\rho_1\left\{1-\frac{4(M_1^2-1)-(\gamma+1)M_1^4}{(\gamma+1)M_1^4\sqrt{M_1^2-1}}\left(\frac{\lambda}{2b}-\sqrt{M_1^2-1}\right)\right\}. \quad (8.2)$$

When σ tends to zero, it follows that

$$\frac{2\Delta}{\lambda}\to 0\,,\qquad Q\to bq_1\rho_1\,,\qquad \frac{\lambda}{2b}\to\sqrt{M_1^2-1}\,,$$

which are the well-known results of Prandtl.

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On the Detached Shock Wave in front of a Body moving at Speeds greater than That of Sound.

By

T. Kawamura

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1. Introduction and Summary

As is well known, when a body like a bullet moves through the air at speeds greater than that of sound, a shock wave is formed in front of the body, and if the velocity of the body is constant, the position of the wave relative to the body remains unaltered. The formation of such a shock wave implies that there is an abrupt change of density and velocity. Ahead of this surface of discontinuity the flow of air is stationary and behind it there is a continuous field of fluid flow which may contain further shock waves.

The equations governing the propagation of the shock wave were established by Rankine (**1**) and his equations determine the relationship between the conditions both in front of and behind a plane shock wave. Meyer has developed Rankine's relations (**2**) and studied the flow in the neighbourhood of an inclined plane or a wedge moving at high supersonic speeds. Later, his analysis was reproduced by Ackeret (**3**), who gave a photograph of the flow in the neighbourhood of a wedge showing that the régime postulated by Meyer in which a shock wave attaches the tip of the body does in fact occur. The limitations of the solutions given by these writers were studied by Bourquard and G. I. Taylor (**4**). Taylor has shown that for a given two-dimensional bullet with semi-vertical angle less than a certain critical value, the necessary condition for the existence of Meyer's régime is that the Mach number of the undisturbed stream should exceed a certain critical value. He has obtained a series of these critical Mach numbers for two-dimensional bullets with various semi-vertical angles, and gave a curve showing the relationship between the semi-vertical angle of the bullet and the said critical Mach number.

Recent developments of the high-speed aeroplanes have stimulated the studies on supersonic aerodynamics and various important investigations have been carried out. Almost all of them, however, are concerned with Meyer's régime, and owing to great mathematical difficulties, no complete theoretical investigations have yet been made so far on the régime in which a detached shock wave is formed in front of an obstacle.

The relationship between the stagnation pressure at the nose of a body and the undisturbed flow ahead of the normal shock wave was discussed by Lord Rayleigh (5) and his result is widely used in experimental researches.

Some time ago, interesting wind tunnel experiments and shooting tests were made by M. Sugimoto (6) and several excellent photographs showing detached shock wave in front of a circular cylinder and a sphere were obtained. Some of them are reproduced on Plate 1 of the present paper for the sake of reference. He also deduced a formula of somewhat empirical nature to locate the detached shock waves, by assuming that the flow behind the shock is equivalent to the subsonic flow of an incompressible fluid subject to appropriate boundary conditions. Thus, assuming a usual potential flow for the subsonic region, he calculated the pressure distribution on the surface of the body, and then comparing the values thus calculated with his own experimental data, he determined the Mach number of the subsonic flow in such a way that the point at which the local speed of sound is first attained is the same in the two cases. Thus, the velocity or pressure distribution in the subsonic region was uniquely determined, and the location of the shock wave was then determined by Rankine's shock wave equations. However, Sugimoto's formula requires empirical data and therefore the scope of its application is limited.

In a recent note, E. V. Laitone (7) has also attacked the problem. He has assumed that the flow behind a shock wave is a part of an incompressible potential flow with the appropriate boundary conditions. The undisturbed velocity of such a potential flow has been taken to be equal to that of the actual stream, while the flow pattern has been assumed to be the same as the potential flow pattern. This assumption determines the complete velocity distributions in the field of flow, while Rankine's relations determine the velocity drop immediately behind the normal portion of the shock wave for a given Mach number of the actual stream, and these conditions locate the normal portion of the detached shock wave. In Laitone's theory no empirical data are needed, but

the assumed flow is not self-consistent, because the flow pattern is assumed to be the same as in the case of subsonic potential flow, notwithstanding that in conformity with the state of affairs in the actual flow the fluid flows with a supersonic speed at infinity upstream. These drawbacks restrict the scope of application of Laitone's theory, and especially it cannot be applied to the case when the Mach number of the actual flow is considerably great.

Quite recently, K. Tamada (8) has also discussed a detached shock wave in front of a circular cylinder and a sphere. He has assumed that a curved shock wave is composed of an infinite number of infinitesimal line-segments and that Rankine's relations should be satisfied at any portion of the curved shock wave. Generally speaking, the curved shock wave will give rise to different amounts of deflection to different streamlines as they pass through it and consequently a certain amount of vorticity will necessarily be generated behind the shock wave. In the close vicinity of the normal portion of the shock wave, however, deflections of the streamlines are small in magnitude and the generated vorticity may be small, too. Thus, if we confine our attentions to the close neighbourhood of the normal portion, the subsonic region behind the shock wave may be approximated by a potential flow. In Tamada's analysis, no unreasonable assumptions have been made on the flow behind the shock wave, because the Mach number of the undisturbed subsonic flow could be determined by means of Rankine's relations.

In the present paper an attempt is made to the discussion of a detached shock wave in front of a body, such as a circular cylinder, a sphere, and a two-dimensional bullet from somewhat different point of view. The flow behind the detached shock wave is approximated by a potential flow as in the papers of previous writers and the effect of the curvature of the shock is considered. In accordance with the results of observations, the shape of the detached shock wave is approximated by a kind of quadratic curve.

Strictly speaking, the application of the present theory is also restricted to the close vicinity of the normal portion, because the equation of continuity is not satisfied exactly. Nevertheless the numerical values given by the present theory are in fairly good agreement with the experimental evidences, and some interesting results are also obtained. Comparing the calculated shape of the detached shock waves in front of a circular cylinder and a sphere with the corresponding observed shock waves it is found that the present theory gives a satisfactory approximation to the actual state of affairs.

The theory is then extended to locate a detached shock wave in front of a two-dimensional bullet. The boundary conditions of the bullet are approximated by Bobyleff's discontinuous flow impinging symmetrically upon a bent lamina. Thus, the head of the bullet is approximated by the bent lamina, while its parallel portion is approximated by the region bounded by two free streamlines starting from the ends of the lamina.

It is found that when the semi-vertical angle of the bullet is greater than $67^\circ 48.9'$, the shock wave can never attach the nose of the bullet but remains always detached for any value of the Mach number of the undisturbed stream. For a bullet with smaller semi-vertical angle than $67^\circ 48.9'$, a detached shock wave is formed in front of the bullet, if the Mach number of the oncoming stream is smaller than a certain critical value, which itself depends upon the value of the semi-vertical angle of the bullet. These results are in good qualitative agreement with Taylor's results.

The present problem has been suggested by Professor S. Tomotika, to whom the writer wishes to express his cordial thanks for his suggestion and continued interest throughout the work. The writer's thanks are also due to Dr. K. Tamada for his invaluable discussions.

2. Assumptions

Referring to Fig. 1, let U be the velocity of the oncoming undisturbed stream and let q and (u, v) be the magnitude of the velocity at any point and its rectangular components respectively. We denote the pressure, the density of the fluid and the velocity of sound in the undisturbed stream by p_1, ρ_1 and c_1 respectively. As is well known, these three quantities are connected by the relation $c_1^2 = \gamma p_1/\rho_1$ where γ is the ratio of the two specific heats of the fluid and takes the value 1.405 for air. Also, we denote by p_2 and ρ_2 the pressure and the density of the fluid immediately behind the shock wave, and by p_0, ρ_0 the pressure and the density at zero velocity for the stream defined by (U, p_1, ρ_1). Further, let α denote the angle between the normal to the shock wave and the streamlines ahead of it and let α denote the angle of deflection from the undisturbed stream of the streamlines immediately behind the shock wave.

In the present paper, a simple assumption is made to the solution of the equation of fluid motion near a solid body. We shall confine our attentions to the case when the body has an axis of symmetry and

the direction of the undisturbed stream is parallel to it. In this case, a shock wave formed in front of the body, either attached or detached, is symmetrical with respect to the axis of symmetry. A particular streamline coinciding with the axis of symmetry experiences no deflection, while the deflection of streamlines from the undisturbed stream becomes greater as the inclination of the shock wave becomes greater. As a consequence of the deflection of streamlines, certain amount of vorticities will be generated in reality behind the curved shock wave and the flow behind the shock wave is no more irrotational. But, if we confine our attentions to the close vicinity of the axis of symmetry, the deflection of streamlines from the undisturbed stream is small and the generated vorticity behind the shock wave may be neglected, and therefore, the flow behind the curved shock wave may be approximated by a potential flow subject to suitable boundary conditions.

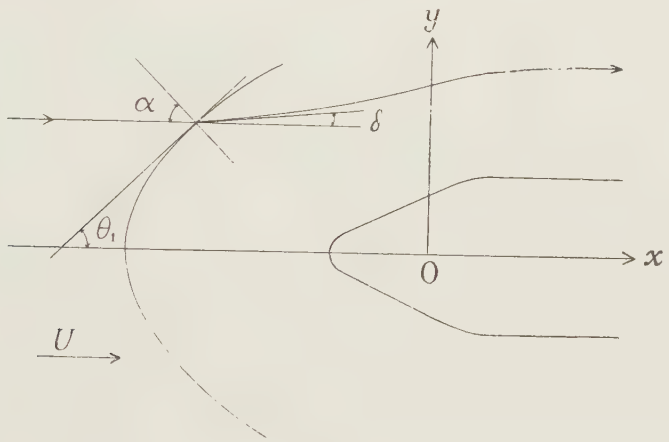


Fig. 1.

If, then, the velocity potential for such an irrotational flow be determined, the velocity distributions behind the shock wave is uniquely determined, and the angle of deflection from the undisturbed stream of the streamlines immediately behind the shock wave is determined by the relation:

$$\tan \delta = v/u . \qquad (1)$$

3. The shock wave equations

Meyer's equations governing the conditions on the two sides of an oblique shock wave are given by

$$\cos^2\alpha = \frac{\{(\gamma-1)+(\gamma+1)\,p_2/p_1\}\,(\gamma-1)}{4\gamma\{(p_0/p_1)^{(\gamma-1)/\gamma}-1\}}, \qquad (2)$$

$$\tan(\alpha+\delta) = \frac{(\gamma-1)+(\gamma+1)\,p_2/p_1}{(\gamma-1)\,p_2/p_1+(\gamma+1)}\tan\alpha\,. \qquad (3)$$

If we denote the Mach number of the undisturbed stream M so that $M = U/c_1$, Bernoulli's theorem gives immediately

$$M^2 = \frac{2}{\gamma-1}\left\{\left(\frac{p_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}}-1\right\}. \qquad (4)$$

Making use of this and eliminating α and p_2/p_1 from the above two equations (2) and (3), we obtain an equation for determining the angle of deflection, δ, of the streamlines as they pass through the shock wave, namely:

$$\tan\delta = \frac{2}{\tan\theta_1}\,\frac{(M^2-1)\tan^2\theta_1-1}{(\gamma-1)\,M^2\tan^2\theta_1+(\gamma+1)\,M^2+2+2\tan^2\theta_1}, \qquad (5)$$

where θ_1 is the angle which a tangent to the shock wave on its arbitrary point makes with the direction of the undisturbed stream and is evidently connected with α by the relation $\theta_1+\alpha=\pi/2$.

Our present purpose is to find out whether a curved shock wave in front of an obstacle is capable of changing the uniform stream into a potential flow whose direction immediately behind the wave is given by equation (1).

In conformity with the results of observations, the detached shock wave ahead of a body can be approximated by a surface of simple form. Since the shape of the section of the detached shock wave by a plane containing the axis of symmetry is similar to a parabola, we shall assume that it can be expressed as:

$$y^2 = a\,(x+t)\,, \qquad (6)$$

where the coordinates x, y have been chosen as shown in Fig. 1. The two parameters a and t in this equation are determined by Meyer's equations (2) and (3) above referred to. As will readily be seen, the latter parameter t represents the distance of the normal portion of the shock wave from the origin of the coordinate-axes.

The coordinates of the normal portion of the shock wave at the intersection of the wave and the axis of symmetry are given by $(-t, 0)$. In the close vicinity of this normal portion, the coordinates of a point

on the shock wave are $(-t + \Delta x, \Delta y)$. A simple calculation gives immediately

$$\tan\theta_1 = \frac{a}{2\Delta y}. \tag{7}$$

Substituting this in equation (5), $\tan\delta$ can be expressed in the form of a series in powers of Δy as:

$$\tan\delta = \frac{4(M^2-1)\Delta y}{a\{(\gamma-1)M^2+2\}}\left\{1 - \frac{4(\gamma+1)M^4}{a^2(M^2-1)\{(\gamma-1)M^2+2\}}(\Delta y)^2 + O[(\Delta y)^4]\right\}. \tag{8}$$

Equation (1) can also be expressed in the form of a series similar to (8), but the coefficients in the series are different according to the form of the obstacle. Comparing the two series expansions for $\tan\delta$ and equating the coefficients of Δy and $(\Delta y)^3$, we obtain two equations for determining the two parameters a and t in equation (6).

A constant factor U_1 in the expression for the velocity potential is determined such that another Rankine's relation connecting the magnitude of the velocities in front of and immediately behind the shock wave is satisfied. If we denote by q_2 the magnitude of the velocity immediately behind the shock wave, the said Rankine's relation may be written as:

$$\frac{q_2}{U} = \frac{\sin\theta_1}{\sin(\theta_1-\delta)}\frac{(\gamma-1)p_2/p_1 + (\gamma+1)}{(\gamma+1)p_2/p_1 + (\gamma-1)}, \tag{9}$$

with

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}M^2\sin^2\theta_1 - \frac{\gamma-1}{\gamma+1}.$$

At the normal portion of the shock wave we have

$$\sin\theta_1 = \sin(\theta_1 - 1) = 1$$

and equation (9) is reduced to

$$\frac{q_2}{U} = \frac{1}{\gamma+1}\left\{(\gamma-1) + \frac{2}{M^2}\right\}, \tag{10}$$

which is identical with one of the conditions used by Laitone (7).

The practical examples will be illustrated in the following sections.

4. The detached shock wave in front of a circular cylinder

We choose the coordinate-axes as shown in the inset of Fig. 2. For convenience, we take the radius of the circular cylinder as unity. The velocity potential ϕ for the continuous irrotational flow past the circular cylinder is given by

$$\phi = -U_1\left(r + \frac{1}{r}\right)\frac{x}{r}, \tag{11}$$

where $r^2 = x^2 + y^2$.

The rectangular components of velocity at any point are given by

$$u = U_1\left\{1 - \frac{x^2 - y^2}{(x^2 + y^2)^2}\right\}, \quad v = -U_1\frac{2xy}{(x^2 + y^2)^2}. \tag{12}$$

Substituting these in (1) we have

$$\tan\delta = \frac{-2xy}{(x^2 + y^2)^2 - x^2 + y^2}. \tag{13}$$

At a point $((\Delta y)^2/a - t, \Delta y)$ on the shock wave near its normal portion, $\tan\delta$ can be expanded in powers of Δy in the form:

$$\tan\delta = \frac{2\Delta y}{t^2(t^2 - 1)}\left\{1 - \frac{(2t^2 + 1)a - 3t^2 + 1}{at^2(t^2 - 1)}(\Delta y)^2 + O[(\Delta y)^4]\right\}, \tag{14}$$

Comparing this with (8) and equating the coefficients of Δy and $(\Delta y)^3$, we obtain, after some calculations,

$$a = \frac{2(M^2 - 1)t(t^2 - 1)}{(\gamma - 1)M^2 + 2} = \frac{t}{2t^2 + 1}\left\{\frac{2(\gamma + 1)M^4}{(M^2 - 1)^2} + 3t^2 - 1\right\}, \tag{15}$$

and these two equations determine the values of the two parameters a and t.

Eliminating a from these equations, we get

$$t^2 = \frac{(3\gamma - 1)M^2 + 4}{8(M^2 - 1)} + \frac{1}{8(M^2 - 1)^2}\left\{(41\gamma^2 - 22\gamma + 17)M^6 - 3(3\gamma^2 - 42\gamma + 35)M^4 - 40(\gamma - 5)M^2 - 80\right\}^{\frac{1}{2}}, \tag{16}$$

and by this equation we can calculate the value of the distance of the normal portion of the shock wave from the centre of the circular cylinder.

If the value of t thus determined is substituted in either of the two equations (15), the value of a can be found, and further, if use is made of the value of a thus obtained, the whole shape of the detached shock wave can be found.

Before proceeding to the numerical discussions, we shall consider two limiting cases. In the first place, when the Mach number, M, of the undisturbed stream increases to infinity, equations (15) are reduced to

$$a = \frac{2t(t^2-1)}{\gamma-1} = \frac{t(3t^2+2\gamma+1)}{2t^2+1}, \tag{17}$$

and equation (16) takes a simple form as:

$$t^2 = \frac{3\gamma-1}{8} + \frac{1}{8}(41\gamma^2 - 22\gamma + 17)^{\frac{1}{2}}, \tag{18}$$

which gives the minimum value of t.

In the case of flow of the air for which $\gamma = 1.405$, it can readily be shown that this minimum value of t is greater than unity, and this implies that even when the Mach number increases to infinity, the shock wave does never attach the circular cylinder and remains always detached.

In the other limiting case when the Mach number M of the undisturbed stream tends to unity, equations (15) give immediately the results that $a = 0$ and $t \to \infty$. Thus, in this case the detached shock wave recedes to infinity upstream and its shape becomes a straight line perpendicular to the direction of the undisturbed stream.

Numerical values of the distance, t, of the normal portion of the detached shock wave from the centre of the cylinder have been calculated by the formula (16) for various values of M and are shown in Table I, where Sugimoto's experimental values and Laitone's theoretical values are also given for comparison. Laitone's values have been calculated by the present writer with the aid of his formula. The results are also shown in Fig. 2, where a dotted-line curve gives Laitone's theoretical values and the crosses show Sugimoto's experimental values. It will be seen that the present theory gives a better approximation to the results of observation than Laitone's theory.

Fig. 3 shows the shape of the detached shock wave in two cases in which M is equal to 2.04 and 2.83 respectively. Observed shock waves

are shown by dotted-line curve for comparison (cf. Plate 1). It is to be noted that although the approximation of the present theory is fairy good only in the close vicinity of the normal portion of the shock wave

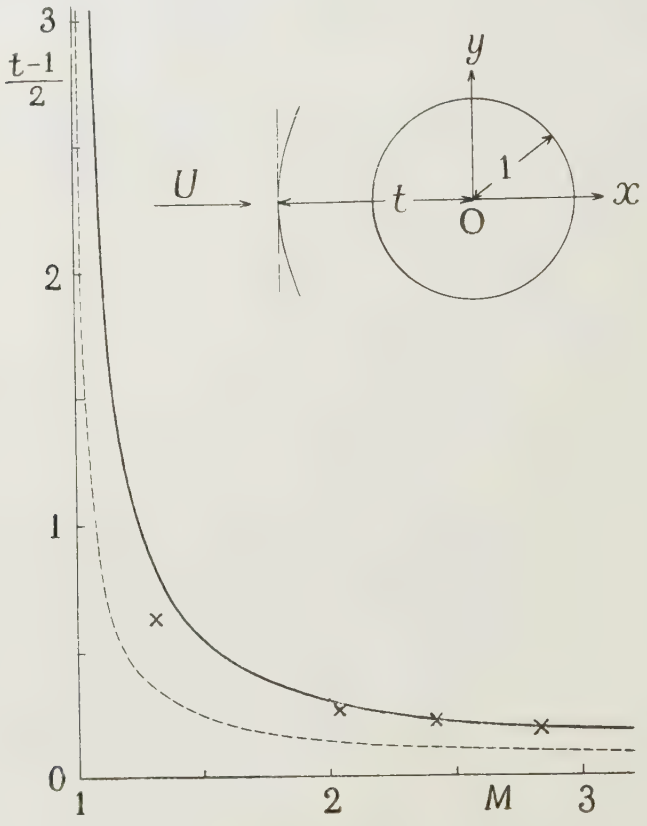
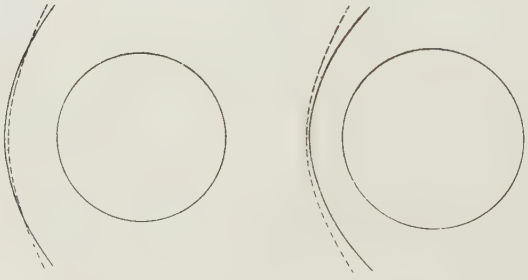


Fig. 2. Case of a circular cylinder. —— Present theory
····· Laitone's theory
× Experiments (Sugimoto)



M=2.04 M=2.83

Fig. 3. Case of a circular cylinder.

but not so good at a finite distance from the normal portion, yet the agreement between the results of the present theory and those of observation is fairy satisfactory.

TABLE I. Values of $(t-1)/2$

Mach number	1.00	1.10	1.34	2.04	2.41	2.83	3.00	∞
Present theory	∞	2.099	0.760	0.283	0.221	0.182	0.172	0.097
Laitone's theory	∞	0.816	0.324	0.129	0.103	0.086	0.082	0.048
Sugimoto's experiments	——	——	0.622	0.261	0.219	0.191	——	——

5. The detached shock wave in front of a sphere

In this section a similar discussion will be made on the location of a detached shock wave in front of a sphere. Taking the centre of the sphere of unit radius as the origin of the coordinate-axes, let the cylindrical coordinates be denoted by x, y and ϑ, instead of usual z, r and θ, the axis of x being taken parallel to the positive direction of the undisturbed stream as shown in the inset of Fig. 4.

In this case the flow is evidently symmetrical about the axis of x, so that the flow pattern is independent of ϑ and it is sufficient to discuss the matter in the xy-plane only. Since the shape of a detached shock wave is approximated by a paraboloid of revolution about the axis of x and hence the section of the wave in the xy-plane is approximated by equation (6).

The velocity potential for the continuous irrotational flow of an incompressible fluid past a sphere is given by

$$\phi = -U_1\left(r + \frac{1}{2r^2}\right)\frac{x}{r}, \tag{19}$$

where $r^2 = x^2 + y^2$.

The components of velocity at any point are

$$u = \frac{U_1}{2r^5}(2r^5 - 2x^2 + y^2), \quad v = -\frac{3U_1}{2r^5}xy, \tag{20}$$

and therefore the angle of deflection, δ, of a streamline at any point from the undisturbed stream is given by

$$\tan\delta = -\frac{3xy}{2r^5 - 2x^2 + y^2}. \tag{21}$$

When the point under consideration lies on the shock wave, x and y must satisfy the relation (6). In the close vicinity of the normal portion of the shock wave, $\tan\delta$ can be expanded in a power series of Δy in the form:

$$\tan\delta = \frac{3\,\Delta y}{2t(t^3-1)}\left\{1 - \frac{(5t^3+1)\,a - 8t^4 + 2t}{2at^2(t^3-1)}(\Delta y)^2 + O[(\Delta y)^4]\right\}. \quad (22)$$

Thus, comparing the two expansions (8) and (22) for $\tan\delta$ and equating the coefficients of Δy and $(\Delta y)^3$, we get, after some calculations,

$$a = \frac{8(M^2-1)t\,(t^3-1)}{3\{(\gamma-1)M^2+2\}} = \frac{t}{5t^3+1}\left\{\frac{3(\gamma+1)M^4}{(M^2-1)^2} + 8t^3 - 2\right\}. \quad (23)$$

Eliminating a from these equations, we have

$$t^3 = \frac{(3\gamma+1)M^2+2}{10(M^2-1)} + \frac{3}{10\sqrt{2}\,(M^2-1)^{\frac{3}{2}}}\{(7\gamma^2-2\gamma+3)M^6 - 2(\gamma-1)(\gamma-8)M^4 - 6(\gamma-5)M^2 - 12\}^{\frac{1}{2}}, \quad (24)$$

and the distance, t, of the normal portion of the shock wave from the centre of the sphere can be calculated by this equation.

In a limiting case when the Mach number of the undisturbed stream tends to infinity, equations (23) and (24) are reduced respectively to

$$a = \frac{8t(t^3-1)}{3(\gamma-1)} = \frac{t(8t^3+3\gamma+1)}{5t^3+1}, \quad (25)$$

and

$$t^3 = \frac{3\gamma+1}{10} + \frac{3}{10\sqrt{2}}(7\gamma^2-2\gamma+3)^{\frac{1}{2}}, \quad (26)$$

the latter giving the minimum value of t.

In the case of flow of the air for which the value of γ is 1.405, the minimum value of t as given by (26) is easily found to be larger than unity. Thus, it is found that as in the case of a circular cylinder, the detached shock wave is always formed in front of the sphere for any value of the Mach number M of the undisturbed stream.

Numerical values of the distance, t, of the normal portion of the detached shock wave have been calculated and are shown in Table II.

The similar values have also been calculated on the basis of Laitone's theory, and they, together with Sugimoto's experimental values, are shown in the table for comparison.

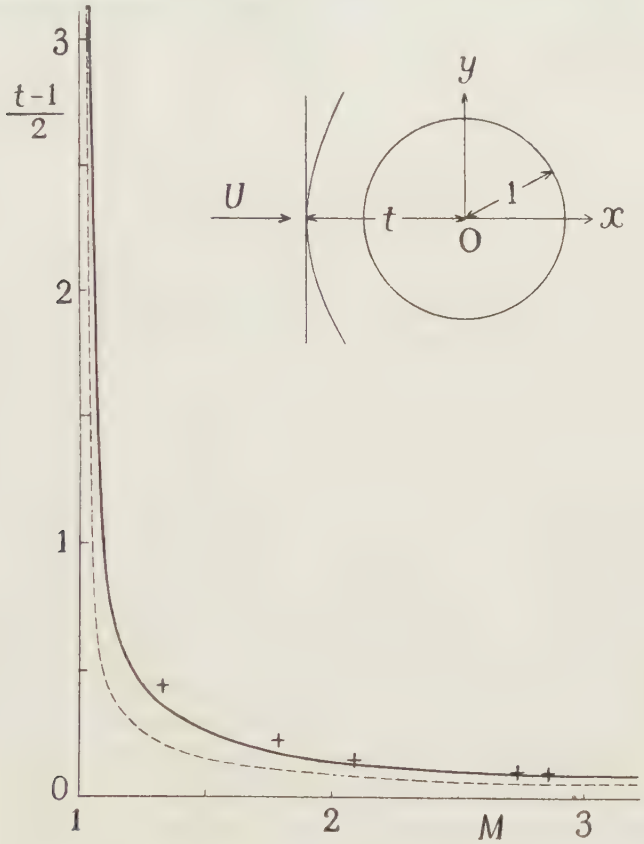


Fig. 4. Case of a sphere. —— Present theory
····· Laitone's theory
\+ Experiments (Sugimoto)

TABLE II. Value of $(t-1)/2$.

Mach number	1.00	1.10	1.32	1.78	2.08	2.40	2.83	∞
Present theory	∞	0.835	0.359	0.171	0.131	0.107	0.088	0.048
Laitone's theory	∞	0.453	0.207	0.103	0.080	0.067	0.056	0.032
Sugimoto's experiments	——	——	0.433	0.206	0.150	0.130	0.103	——

The results are also shown in Fig. 4. It will be seen that the results of the present theory are in good agreement with those of Sugimoto's observation.

Making use of the value of a as calculated by either of the two equations (23), the form of the detached shock wave has been calculated in two cases in which the value of the Mach number M is 2.08 and 2.63 respectively. They are shown in Fig. 5, where the dotted curves indicate the location of the shock waves found experimentally.

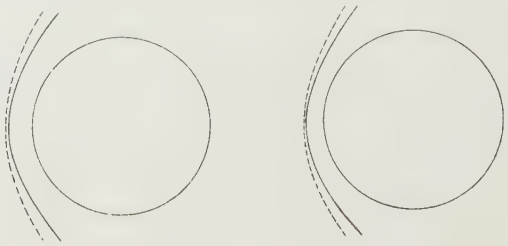


Fig. 5. Case of a sphere.

6. The shock wave in front of a two-dimensional bullet

In the present section the writer will discuss a detached shock wave in front of a two-dimensional bullet. The similar problem has been discussed by Laitone and Taylor. Laitone has approximated a bullet by an appropriate distribution of line sources along the axis of the bullet, while Taylor has approximated a bullet by a two-dimensional wedge.

The writer now approximates the tip portion of a two-dimensional bullet by a bent lamina and the parallel portion of it by a pair of free streamlines starting from the two edges of the lamina and extending to infinity downstream. Such a discontinuous flow past a bent lamina was discussed by Bobyleff as early as 1881 and short account of his theory is given several text-books on hydrodynamics (**9**). Recently, a detailed discussion on Bobyleff's problem has been made by S. Tomotika and Z. Hasimoto (**10**). Before proceeding further, we shall recapitulate here the essential part of Bobyleff's theory for the sake of later use.

We assume that a uniform flow impinges symmetrically upon a bent lamina $A_1C A_2$ whose section consists of two equal straight lines forming an angle (Fig. 6 (a)). Let 2β be the angle measured on the downstream side, and also let λ_1 and λ_2 denote a pair of free streamlines starting from the two edges A_1 and A_2 of the lamina and extending to infinity downstream. Taking the plane of fluid motion as the z-plane, we take the x- and y-axes as shown in Fig. 6 (a).

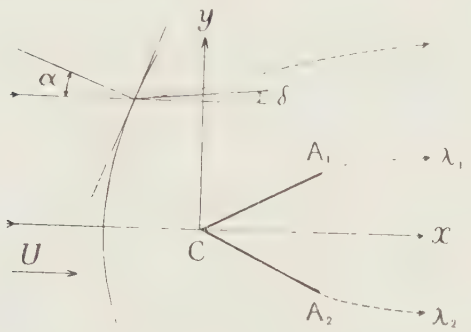


Fig. 6. (a) z-plane.

Let $f = \phi + i\psi$ be the complex velocity potential for the discontinuous flow under consideration. The f-plane is shown in Fig. 6 (b).

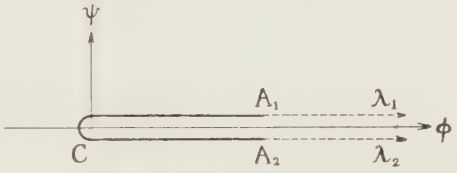


Fig. 6. (b) f-plane.

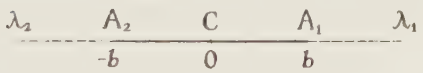


Fig. 6. (c) s-plane.

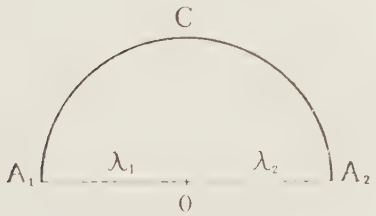


Fig. 6. (d) ζ-plane.

This f-plane is transformed into the upper-half of an s-plane by the transformation:

$$f = s^2 . \tag{27}$$

The two points A_1 and A_2 corresponding to the two edges of the lamina are then transformed into the points on the real axis of the s-plane whose coordinates are $(b, 0)$ and $(-b, 0)$ respectively.

The upper-half of the s-plane is transformed into the interior of a semi-circle of unit radius in a ζ-plane by the equation:

$$s = -\frac{b}{2}\left(\zeta + \frac{1}{\zeta}\right). \tag{28}$$

The bent lamina is transformed into the circumference of the semi-circle, while the free streamlines are transformed into a diameter which coincides with the real axis of the ζ-plane.

The conjugate complex velocity at any point in the flow is given by

$$\frac{df}{dz} = e^{-i\Omega}, \tag{29}$$

with

$$\Omega = \delta + i \log q, \tag{30}$$

where δ denotes the inclination to the axis of x of a streamline at any point and q is the magnitude of the velocity there.

If we can find Ω as a function of ζ, we can completely determine df/dz or the velocity distributions in the stream. The conditions which should be satisfied by the function $\Omega(\zeta)$ are as follows: (i) it must be an analytic function of ζ regular in the interior of the semi-circle $A_1C\,A_2$; (ii) on the circumference of the semi-circle the real part of $\Omega(\zeta)$ must satisfy the conditions:

$$\left.\begin{aligned} \Re[\Omega(\zeta)] &= -\beta \quad (0 < \vartheta < \pi/2), \\ &= \beta \quad (\pi/2 < \vartheta < \pi); \end{aligned}\right\} \tag{31}$$

and lastly, (iii) the imaginary part of $\Omega(\zeta)$ must vanish on the diameter $A_1C\,A_2$.

Such a function can be determined by Poisson's formula. Omitting the details of analysis for brevity, the final results only will be given. We have

$$\Omega(\zeta) = \frac{2\beta}{i\pi} \log \frac{i+\zeta}{i-\zeta}. \tag{32}$$

If we eliminate s and f from equations (27), (28), (29) and (32), we obtain

$$dz = \frac{b^2}{2} \left\{\frac{i+\zeta}{i-\zeta}\right\}^{2\beta/\pi} \left(\zeta^2 - \frac{1}{\zeta^2}\right) \frac{1}{\zeta} d\zeta, \tag{33}$$

and integration of this equation would give z as a function of ζ. Since, however, the analytical integration is almost hopeless, we have carried out numerical integration.

The half-length L of the bent lamina is given by

$$L = \mathrm{A_1C} = 2b^2\left\{\frac{1}{2} + \frac{\beta}{\pi} + \frac{\beta^2}{\pi^2}\psi\left(\frac{2-\beta/\pi}{2}\right) - \frac{\beta^2}{\pi^2}\psi\left(\frac{1-\beta/\pi}{2}\right)\right\}, \tag{34}$$

where $\psi(x) = \dfrac{d}{dx}\log\Gamma(x)$. When the value of L is given, the value of the parameter b in equation (28) can be determined by equation (34).

Next, we shall proceed to the discussion of the location of a detached shock wave in front of a two-dimensional bullet, by applying the above results. In the close vicinity of the axis of x, the direction of a streamline is approximated as:

$$\tan\delta = \delta + \frac{1}{3}\delta^3 + \cdots\cdots, \tag{35}$$

with

$$\delta = \Re\left[\Omega(\zeta)\right]. \tag{36}$$

Referring to the coordinate-axes in Fig. 6 (a), let the coordinates of the normal portion of the shock wave be denoted by $(-t, 0)$. In the close vicinity of this normal portion, the coordinates of a point on the shock wave are given by $(-t + \Delta x, \Delta y)$, and since x and y must satisfy (6), $\Delta x = (\Delta y)^2/a$. Thus, if we write $\Delta z = \Delta x + i\Delta y$, we have

$$\Delta z = \frac{1}{a}(\Delta y)^2 + i\Delta y. \tag{37}$$

It is readily found that the coordinates of a point in the ζ-plane which corresponds to the normal portion of the shock wave are given by $(0, i\eta_1)$. The increment $\Delta\zeta$ corresponding to Δz given by (37) is

$$\begin{aligned}\Delta\zeta &= \left(\frac{d\zeta}{dz}\right)_{i\eta_1}\Delta z + \frac{1}{2}\left(\frac{d^2\zeta}{dz^2}\right)_{i\eta_1}(\Delta z)^2 + \frac{1}{6}\left(\frac{d^3\zeta}{dz^3}\right)_{i\eta_1}(\Delta z)^3 + \cdots\cdots \\ &= \left(\frac{d\zeta}{dz}\right)_{i\eta_1}\left[\Delta z + \frac{1}{2}\left(\frac{d}{d\zeta}\frac{d\zeta}{dz}\right)_{i\eta_1}(\Delta z)^2 + \frac{1}{6}\left\{\left(\frac{d^2}{d\zeta^2}\frac{d\zeta}{dz}\right)_{i\eta_1}\left(\frac{d\zeta}{dz}\right)_{i\eta_1} + \left(\frac{d}{d\zeta}\frac{d\zeta}{dz}\right)^2_{i\eta_1}\right\}(\Delta z)^3 + \cdots\cdots\right].\end{aligned} \tag{38}$$

The inclination δ of a streamline passing through a point in the close vicinity of the normal portion is given, by means of (36), in the form:

$$\delta = \Re\left\{\left(\frac{d\Omega}{d\zeta}\right)_{i\eta_1} \Delta\zeta + \frac{1}{2}\left(\frac{d^2\Omega}{d\zeta^2}\right)_{i\eta_1} (\Delta\zeta)^2 + \frac{1}{6}\left(\frac{d^3\Omega}{d\zeta^3}\right)_{i\eta_1} (\Delta\zeta)^3 + \cdots\cdots\right\}. \quad (39)$$

Thus, if use is made of (37) and (38), we easily obtain the series expansion of δ in powers of Δy, and if the series for δ thus obtained be inserted in the right-hand side of (36), $\tan\delta$ can be expressed in the form of a series in powers of Δy as:

$$\begin{aligned}\tan\delta = i\left(\frac{d\Omega}{d\zeta}\right)_{i\eta_1}\left(\frac{d\zeta}{dz}\right)_{i\eta_1}\Delta y\Bigg[1 + \Bigg\{&\frac{1}{a}\left(\frac{d}{d\zeta}\frac{d\zeta}{dz}\right)_{i\eta_1} - \frac{2i\eta_1}{a(1-\eta_1^2)}\left(\frac{d\zeta}{dz}\right)_{i\eta_1} \\ &+ \frac{1+3\eta_1^2}{3(1-\eta_1^2)^2}\left(\frac{d\zeta}{dz}\right)^2_{i\eta_1} + \frac{i\eta_1}{1-\eta_1^2}\left(\frac{d}{d\zeta}\frac{d\zeta}{dz}\right)_{i\eta_1}\left(\frac{d\zeta}{dz}\right)_{i\eta_1} \\ &- \frac{1}{6}\left(\frac{d^2}{d\zeta^2}\frac{d\zeta}{dz}\right)_{i\eta_1}\left(\frac{d\zeta}{dz}\right)_{i\eta_1} - \frac{1}{6}\left(\frac{d}{d\zeta}\frac{d\zeta}{dz}\right)_{i\eta_1} \\ &- \frac{1}{3}\left(\frac{d\Omega}{d\zeta}\right)^2_{i\eta_1}\left(\frac{d\zeta}{dz}\right)^2_{i\eta_1}\Bigg\}(\Delta y)^2 + \mathrm{O}[(\Delta y)^4]\Bigg]. \quad (40)\end{aligned}$$

Various terms in the coefficients of Δy and $(\Delta y)^3$ can be evaluated immediately. Thus, writing $\sigma = 2\beta/\pi$ for shortness, we have

$$\left(\frac{d\Omega}{d\zeta}\right)_{i\eta_1} = -\frac{2\sigma}{1-\eta_1^2}, \qquad \left(\frac{d\zeta}{dz}\right)_{i\eta_1} = \frac{2i}{b^2}\left(\frac{1-\eta_1}{1+\eta_1}\right)^{\sigma}\frac{\eta^3}{1-\eta_1^4},$$

$$\left(\frac{d}{d\zeta}\frac{d\zeta}{dz}\right)_{i\eta_1} = \frac{2}{b^2}\left(\frac{1-\eta_1}{1+\eta_1}\right)^{\sigma}\frac{\eta_1^2}{1-\eta_1^4}\left(-\frac{2\sigma\eta_1}{1-\eta_1^2} + \frac{3+\eta_1^4}{1-\eta_1^4}\right),$$

$$\begin{aligned}\left(\frac{d^2}{d\zeta^2}\frac{d\zeta}{dz}\right)_{i\eta_1} = &-\frac{2i}{b^2}\left(\frac{1-\eta_1}{1+\eta_1}\right)^{\sigma}\frac{\eta_1}{1-\eta_1^4} \\ &\times\left[\left(-\frac{2\sigma\eta_1}{1-\eta_1^2} + \frac{2(1+\eta_1^4)}{1-\eta_1^4}\right)\left(-\frac{2\sigma\eta_1}{1-\eta_1^2} + \frac{3+\eta_1^4}{1-\eta_1^4}\right)\right. \\ &\left. - \frac{2\sigma\eta_1(1+\eta_1^2)}{(1-\eta_1^2)^2} + \frac{7\eta_1^4}{(1-\eta_1^4)^2}\right]. \quad (41)\end{aligned}$$

Finally, comparing the above series for $\tan\delta$ with (8) and equating the coefficients of Δy and $(\Delta y)^3$, we obtain, after some reductions,

$$a = \frac{M^2-1}{(\gamma-1)M^2+2}\,\frac{b^2}{\sigma}\left\{\left(\frac{1-\eta_1}{1+\eta_1}\right)^{\sigma}\frac{\eta_1^3}{(1-\eta_1^2)(1-\eta_1^4)}\right\}^{-1}, \quad (42)$$

and

$$\frac{16}{3}\frac{(M^2-1)^2}{a^2\{(\gamma-1)M^2+2\}^2}+\frac{4(\gamma+1)M^4}{a^2(M^2-1)\{(\gamma-1)M^2+2\}}$$

$$=-\frac{2}{ab^2}\left(\frac{1-\eta_1}{1+\eta_1}\right)^{\sigma}\frac{\eta_1^2}{(1-\eta_1^4)^2}\left\{3+2\eta_1^2+3\eta_1^4-2\sigma\eta_1(1-\eta_1^2)\right\}$$

$$+\frac{2}{3}\frac{1}{b^2}\left(\frac{1-\eta_1}{1+\eta_1}\right)^{2\sigma}\frac{\eta_1^4}{(1-\eta_1^4)^4}$$

$$\times\left\{15+20\eta_1^2+49\eta_1^4+20\eta_1^6+15\eta_1^8+8\sigma^2\eta_1^2(1+\eta_1^2)^2\right.$$

$$\left.-8\sigma\eta_1(1+\eta_1^2)(3+2\eta_1^2+3\eta_1^4)\right\}. \qquad (43)$$

These two equations determine the values of the two parameters η_1, a.

In the close vicinity of the tip of the bullet, η_1 can be written as:

$$\eta_1=1-\varepsilon, \qquad (44)$$

where ε is an infinitesimal quantity. Equations (42) and (43) are then reduced respectively to

$$a=\frac{8b^2(M^2-1)}{(\gamma-1)M^2+2}\sigma^{-1}\varepsilon^{2-\sigma}2^{\sigma}, \qquad (45)$$

and

$$\frac{24(\gamma M^4+2M^2-1)}{(M^2-1)^2}\sigma^2+\left\{\frac{128(M^2-1)}{(\gamma-1)M^2+2}+48\right\}\sigma$$

$$-\frac{119(M^2-1)}{(\gamma-1)M^2+2}=0. \qquad (46)$$

These two equations determine a certain critical Mach number, M_{crit}, for a bullet with a given semi-vertical angle less than a certain value, such that when the Mach number of the undisturbed stream is greater than M_{crit} the shock wave attaches the tip of the bullet.

If, however, the semi-vertical angle of a bullet becomes greater than a certain critical value, the shock wave does never attach the tip of the bullet for any large value of the Mach number, and there is formed a detached shock wave in front of the bullet. Such a critical semi-vertical angle $\beta_{\max}(=\frac{1}{2}\pi\sigma_{\max})$ can be found by solving an equation to which equation (46) is reduced in the limit $M\to\infty$, namely:

$$24\gamma\sigma_{\max}^2+\left(\frac{128}{\gamma-1}+48\right)\sigma_{\max}-\frac{119}{\gamma-1}=0. \qquad (47)$$

For the case of air ($\gamma = 1.405$), we have found that

$$\sigma_{\max} = 0.7535\,, \qquad \beta_{\max} = 67^\circ 48.9'\,. \tag{48}$$

Numerical values of the critical Mach number M_{crit} for bullets with various values of σ have been calculated and they are tabulated in Table III. Similar values of M_{crit} for wedges are given in Table IV for comparison. These latter values for wedges have been calculated by the present writer with the aid of the following two equations, which have been given by D. C. Pack in his recent paper (**11**).

$$2\tan^2 m \tan^3\alpha + \tan\beta \tan^2\alpha \{(\gamma + 1) + (\gamma + 3)\tan^2 m\} - 2\tan\alpha + \tan\beta \{(\gamma - 1) + (\gamma + 1)\tan^2 m\} = 0\,, \tag{49}$$

and

$$(\gamma + 1)\tan\beta \tan^4\alpha - 4\tan^3\alpha + 2(\gamma - 3)\tan\beta \tan^2\alpha - 4\tan^2\beta \tan\alpha + (\gamma + 1)\tan\beta = 0\,, \tag{50}$$

where $\operatorname{cosec} m = M$ and β denotes the semi-vertical angle of the wedge.

These equations are the analytical expressions for Taylor-Maccoll's analysis (**12**) where they have obtained their results by a graphical method.

TABLE III. Values of M_{crit} for our bullets.

β	2° 7.8′	5° 8.2′	11°24.9′	24°24.9′	29°29.7′	33°44.7′	45°44.3′	67°48.9′
M_{crit}	1.050	1.100	1.200	1.400	1.500	1.600	2.000	∞

TABLE IV. Values of M_{crit} for wedge.

β	5°	10°	15°	20°	30°	40°	45° 23′
M_{crit}	1.240	1.417	1.616	1.843	2.530	4.520	∞

The results for both our bullet and wedge are shown in Fig. 7. It will be seen that our results for the two-dimensional bullet are in good qualitative agreement with those for the wedge.

Numerical values of η_1, which specifies the location of the normal portion of the detached shock wave, have been calculated by the formulae (45) and (46) for various values of M, and the corresponding values of the distance, t, of the normal portion of the shock wave from the tip of the bullet have been obtained by a graphical method.

The results are shown in Fig. 8 in four cases in which the semi-vertical angle β of the bullet is equal to 15°, 30°, 60° and 67° 38.9′ respectively. In this figure are also shown Laitone's values for comparison, which have been calculated by the present writer by means of Laitone's formula. The two series of curves deviate considerably from each other especially in the close vicinity of the horizontal axis.

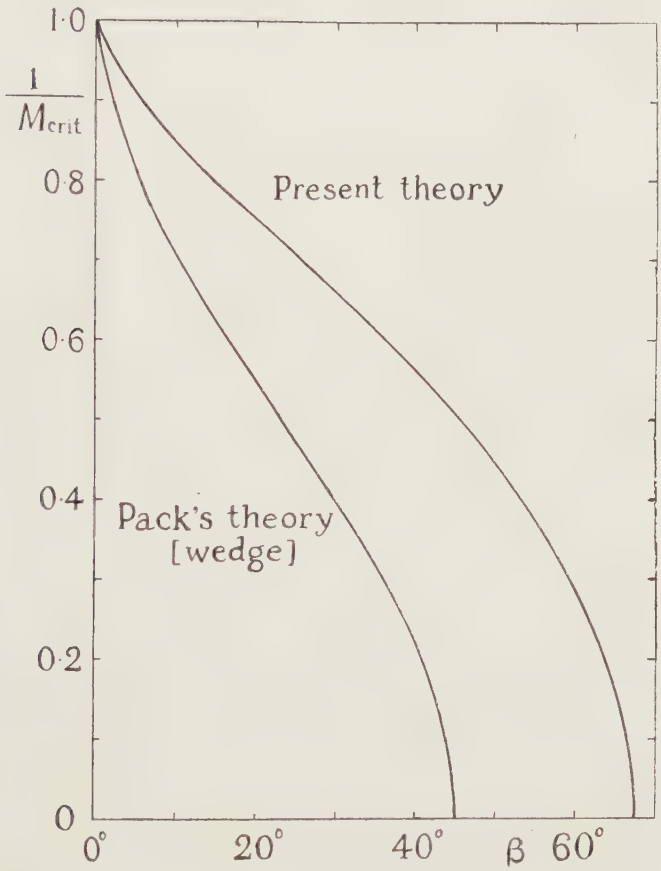


Fig. 7. Values of the critical Mach number plotted against the semi-vertical angle of a bullet or a wedge.

When the semi-vertical angle of the bullet is smaller than 67° 48.9′, each of our curves cuts the horizontal axis at a finite angle. On the other hand, any of Laitone's curves can never touch the horizontal axis and the least value of the distance of the normal portion of the detached shock wave from the tip of the bullet is always different from zero.

This implies evidently that according to his theory the shock wave remains always detached for any values of the semi-vertical angle and of the Mach number of the undisturbed flow. In fact, we cannot determine the critical values of M_{crit} and β_{max} on the basis of Laitone's theory. These results are not only in contradiction to Taylor's theoretical results but also to experimental evidences.

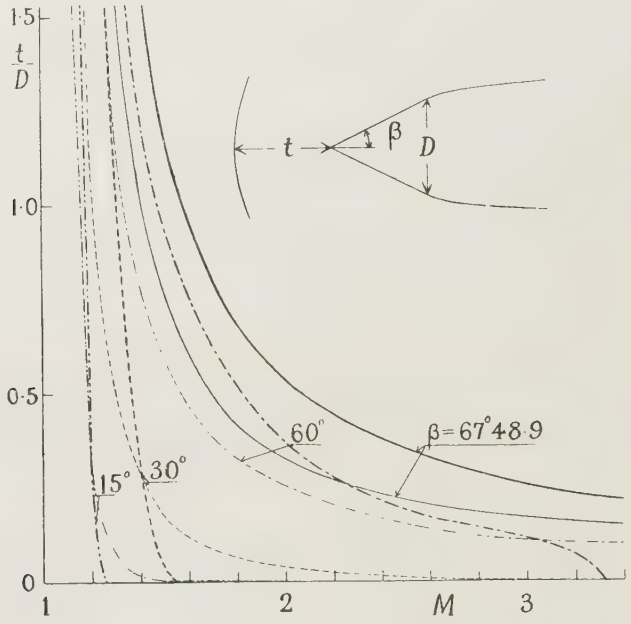


Fig. 8. Case of two-dimensional bullets. Thick-line curves give the values obtained by the present theory, while thin-line curves give the corresponding values obtained by Laitone's theory.

Fig. 9 is a reproduction of a figure in Kármán's recent paper (**13**), which shows the experimental values of the ratio of the distance of the detached shock wave from the tip of a bullet to its diameter as a function of the Mach number of the undisturbed flow. These experimental data have been obtained by shooting tests of a conical headed three-dimensional bullet with semi-vertical angle of 20°.

In Fig. 10 is given a similar curve showing our theoretical results for a two-dimensional bullet with semi-vertical angle of 15°, and this may be compared with Fig. 9. It will be seen that there is a good qualitative agreement between two figures.

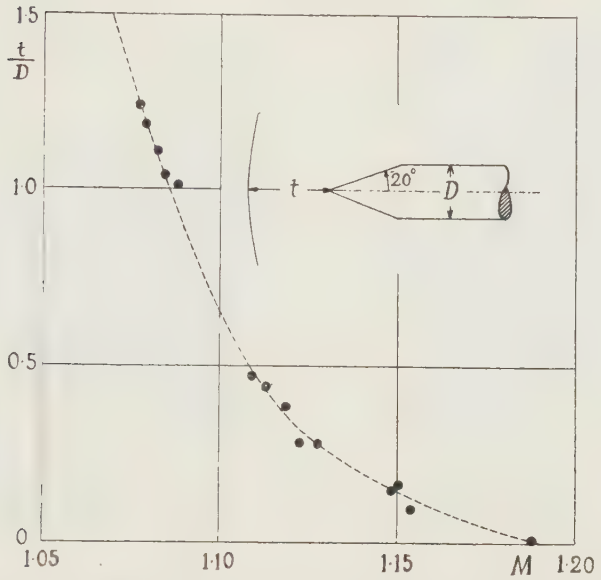


Fig. 9. Experimental values of t/D plotted against M, for a projectile with semi-vertical angle of 20° (Kármán (**13**)).

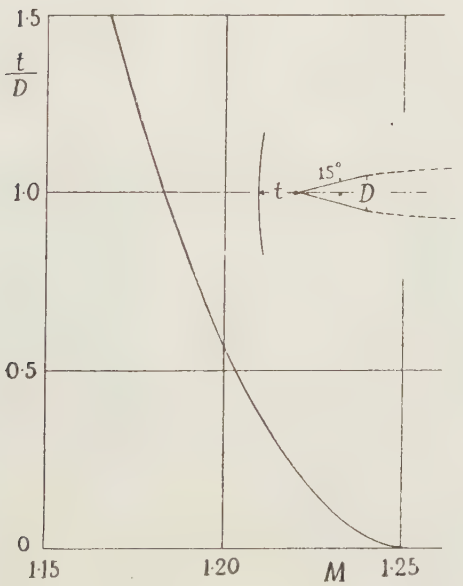


Fig. 10. Calculated values of t/D for a two-dimensional bullet with semi-vertical angle of 15°.

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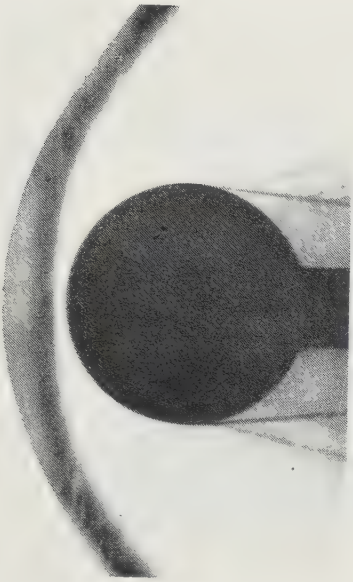
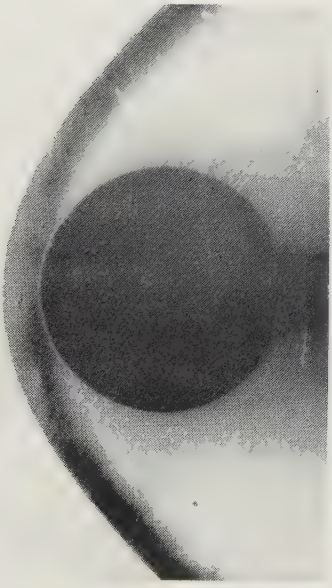


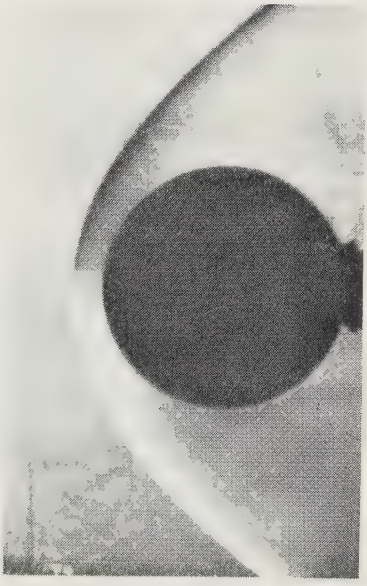
$M=2.04$

$M=2.83$

Circular cylinder

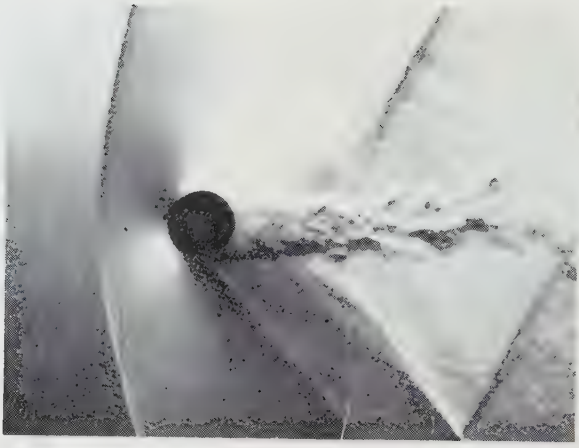


$M=2.08$

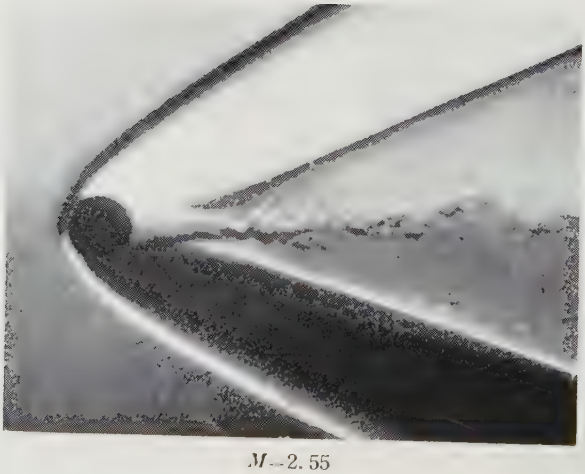


$M=2.63$

Sphere



M=1. 15



M=2. 55

Sphere (shooting test)

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On the Transformation of Spin Functions.

By

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SUMMARY

In order to determine the states of a system having several identical particles, each with any spin, we have essentially to consider the transformation of the spin functions of the system, and to treat the problem three principal methods may be expected. In the present paper we have treated the problem by the method of simultaneous reduction of both representations of rotation and symmetric groups, regarding the spin functions as the bases. The results of our calculation are given in Table I. By adopting these results it has been able to obtain, for example, the transformation of nuclear spin functions of a system having identical nuclei with any spatial symmetry.

1. Introduction

In quantum mechanics, for the purpose of determining the states of a system including several identical particles, we have essentially to find the transformation of the spin functions of the set in relation to the problem of permutation degeneracy and of statistical property subjected to the particle itself. The problem concerning a set of particles with spin $\frac{1}{2}\hbar$, e.g. many-electron problem, has been dealt with by the following methods: firstly, the wave method based upon the employment of explicit wave functions (**1**); secondly, the method of Dirac's character operator (**2**), which has been extended by Van Vleck (**3**), Serber (**4**), and Corson (**5**); and finally, a method which makes use of the representation matrices of the symmetric permutation group given by Serber (**6**) and by Yamanouchi (**7**).

In the present paper we have attempted to find the transformation of spin functions of a system composed of several identical particles, each having a higher spin. And we have treated especially along the line of the last method, performing the simultaneous reduction of both representations of rotation and symmetric groups regarding the spin functions as the bases, without having recourse to the construction of

the representation matrices of the symmetric group. The results of our calculation are given in Table I.

By adopting these results we can obtain the transformation of nuclear spin functions of a system including identical nuclei with any spatial symmetry*. Then, some applications will be possible to the problem not only of nuclear spin degeneracy, nuclear quadrupole coupling, and nuclear spin-spin interaction in molecular and crystal systems, but also of nuclear shell structures.

2. General remarks on the simultaneous reduction of the two group representations

The spin space of a particle with spin s (taking $\hbar$ as unit) is expressed by the representation $\mathfrak{D}_s$ of a rotation group, diagonalizing the rotation about the z-axis in a $(2s+1)$-dimensional unitary space. The spin function of a system including f similar particles are made up of $(2s+1)^f$ independent bases:

$$U_{\lambda}^{(1)}U_{\mu}^{(2)}\cdots U_{\nu}^{(f)}, \text{ where } \lambda, \mu, \cdots, \nu = s, s-1, \cdots, -s. \quad (1)$$

Therefore they make a $(2s+1)^f$-dimensional vector space $\mathfrak{R}$, in which they are linearly transformed not only by rotations of the space, but also by permutations of these identical particles. So we have two representations of the rotation and the symmetric groups in the space $\mathfrak{R}$ at the same time: the former, $[\mathfrak{D}_s]_f$, the Kronecker f-th power of $\mathfrak{D}_s$, and the latter, π_f of degree f. In the reduction of $\mathfrak{R}$ the following theorem is employed: the reduction of either representation can be carried out simultaneously, since the operations of these two groups are commutative with each other. This means, as regards the basis vectors, that it is possible to divide into a set of rectangles with the following form:

$$\begin{matrix} V_{11}, \cdots, V_{1n} & ; & V'_{11}, \cdots, V'_{1n'} & ; & \cdots\cdots \\ \cdots\cdots & & \cdots\cdots & & \\ V_{k1}, \cdots, V_{kn} & & V'_{k'1}, \cdots, V'_{k'n'} & & \end{matrix} \quad (2)$$

Every set of basis vectors belonging to every row in one of the rectangles appertains together to a certain irreducible representation of a group. And in the same way it holds between every column in one of the

* The spatial symmetry of the fixed nuclei is expressed by the *symmetry group* ⑤ (8), which is a finite or continuous sub-group of the three-dimensional full rotation-reflection group.

rectangles and another group representation. Consequently either irreducible representation of the two groups corresponds to one of the rectangles (**10**).

According to the above theorem, to each rectangle corresponds an irreducible representation specified by a certain resultant spin value S of the rotation group and one of the symmetric group, taking suitable linear combinations of (1) as the bases. In our case, however, to all the columns of each rectangle belongs not always an irreducible representation of a group, that is to say, the representation of the symmetric group belonging to a rectangle is not irreducible in general. If the representation of the rotation group has been completely reduced, each rectangle has one-to-one correspondence with each S which is given by the reduction of the product representation:

$$[\mathfrak{D}_s]_f = \sum_S c(S,f)\,\mathfrak{D}_S;\ S = fs - g\,, \tag{3}$$

where $g = 0, 1, 2, \cdots, fs$, or $fs - \frac{1}{2}$ for all the possible resultant spin values S's, and $c(S,f)$ is the number of times of the appearance of an irreducible representation $\mathfrak{D}_S$.

Now we take

$$U_\lambda^{(i)} = \frac{(u_1^{(i)})^{s+\lambda}(u_2^{(i)})^{s-\lambda}}{\sqrt{(s+\lambda)!\,(s-\lambda)!}}, \tag{4}$$

as the unitary base vector of the spin space of each particle, where $i = 1, 2, \cdots, f$ and $\lambda = s, s-1, \cdots, -s$. And if we introduce the contragradient variables x_1 and x_2 to $u_1^{(i)}$ and $u_2^{(i)}$, respectively, the following invariant formula can be established:

$$I = \prod_{i<j}(u_1^{(i)}u_2^{(j)} - u_2^{(i)}u_1^{(j)})^{\sigma_{ij}} \prod_k (u_1^{(k)}x_1 + u_2^{(k)}x_2)^{2s-g_k}\,, \tag{5}$$

where $\prod_{i<j}$ means the product with respect to i less than j, subject to the condition that $\sum_{i<j}\sigma_{ij} = g$, and

$$g_k = \begin{cases} \sum'\sigma_{ij} & (\text{for } k = i, \text{or } j) \\ 0 & (\text{for } k \neq i, \text{and } j)\,, \end{cases}$$

$\sum'$ being the summation with respect to all k's. Then it is easily proved that the basis vector of the representation $\mathfrak{D}_S$ is given for $M = S, S-1, \cdots, -S$ by the coefficient of

$$X_M^S = \frac{x_1^{S+M}\,x_2^{S-M}}{\sqrt{(S+M)!\,(S-M)!}} \tag{6}$$

in the expansion of the invariant formula (5). Thereby we can explicitly obtain each column in every rectangle (2) except for common numerical factor.

From the fact that the characters of the representations in $\mathfrak{R}$ are invariant for any set of linear combinations, those characters of a transformation AP in the spin space can be calculated by the following two methods: first, by the bases (2) reduced only for the rotation group, and secondly, by the basis vectors of the type (1); where A means a set of unitary transformations:

$$U_s = \zeta^s U_s,\ U_{s-1} = \zeta^{s-1} U_{s-1},\ \cdots,\ U_{-s} = \zeta^{-s} U_{-s}, \tag{7}$$

on remembering that the representation $R_z(\varphi)$ of the rotation (angle φ) about the z-axis with the bases $U_s, U_{s-1}, \cdots, U_{-s}$ can be given by

$$R_z(\varphi) = \begin{pmatrix} e^{is\varphi} & & & 0 \\ & e^{i(s-1)\varphi} & & \\ & & \ddots & \\ 0 & & & e^{-is\varphi} \end{pmatrix}.$$

P is a set of permutations which have various cycle structures $(\alpha_1, \alpha_2, \cdots, \alpha_\rho)$, provided that $\alpha_1 + 2\alpha_2 + \cdots + \rho\alpha_\rho = f$ (**9**). Let $\chi_S(P)$ be a set of characters of the representation of the permutations P belonging to a rectangle specified by a given S. Then as both results must be equal, we can find the following identities, namely, for an integral value of S:

$$\sum_S \chi_S(P)\,(\zeta^S + \zeta^{S-1} + \cdots + \zeta^{-S}) = \prod_{i=1}^{\rho} (\zeta^{s\alpha_i} + \zeta^{(s-1)\alpha_i} + \cdots + \zeta^{-s\alpha_i}), \tag{8 a}$$

and for a half-integral value of S:

$$\sum_S \chi_S(P)\,(\zeta^{2S} + \zeta^{(2S-2)} + \cdots + \zeta^{-2S}) = \prod_{i=1}^{\rho} (\zeta^{2s\alpha_i} + \zeta^{(2s-2)\alpha_i} + \cdots + \zeta^{-2s\alpha_i}), \tag{8 b}$$

respectively, where $\sum_S$ is a summation about all possible $S = fs - g \geqq 0$. Thereupon $\chi_S(P)$ can be obtained as the coefficient of ζ^g in a polynomial:

$$\prod_{i=1}^{\rho} (1 - \zeta)\,(\zeta^{2s\alpha_i} + \zeta^{(2s-1)\alpha_i} + \cdots + 1), \tag{9}$$

if both sides of (8 a) and (8 b) are multiplied by $\zeta^{sf}(1 - \zeta)$, and $\zeta^{2sf} \times (1 - \zeta^2)$ for $s =$ an integer and a half-integer, respectively, and ζ^2 is replaced by ζ for the latter.

Thus we can calculate $\chi_S(P)$ about every class of the symmetric group over all possible S. The calculations have been carried out within the limits of the problem of some part of the nuclear spins and of $f \leqq 6$. It is shown that the results are equal to those found from branching diagrams. In addition, a character of the unit element is the degree of dimension of the representation matrix of the symmetric group belonging to a given S, being just equal to $c(S, f)$ in (3).

Finally, the complete reductions of π_f are given by resolving every $\chi_S(P)$ calculated above into simple characters of the symmetric group (9). The final results of the simultaneous reductions are inserted in Table I, where in case of $s = 1$, $f = 6$, and $S = 2$, the result is given by the direct sum:

$$\{6\} + \{5, 1\} + 2\{4, 2\} + \{3, 2, 1\}$$

and every $\{\alpha_1, \cdots, \alpha_\rho\}$ means an irreducible representation of the symmetric group, above all $\{f\}$ is always an identical one. In these illustrations we can find the fact that, if $2s + 1 \geqq f$, every irreducible representation is at least once included in the direct sum.

TABLE I. The results of the simultaneous reduction.

The first row of each sub-table gives the irreducible representations of each symmetric group $\mathfrak{S}_f$, and the first column of them means the resultant spins characterizing the irreducible representations of the rotation group. In case when $f = 6$, χ_i's give $\{6\}$, $\{5,1\}$, $\{4,2\}$, $\{4,1^2\}$, $\{3^2\}$, $\{3,2,1\}$, $\{3,1^3\}$, $\{2^3\}$, $\{2^2,1^2\}$, $\{2,1^4\}$, and $\{1^6\}$ with the order of i. In case when $f = 5$, they give $\{5\}$, $\{4,1\}$, $\{3,2\}$, $\{3,1^2\}$, $\{2^2,1\}$, $\{2,1^3\}$, and $\{1^5\}$, respectively. And in case when $f = 4$, $\chi_2 = \{3,1\}$, $\chi_3 = \{2^2\}$, $\chi_4 = \{2,1^2\}$, and $\chi_5 = \{1^4\}$. In the following table any column of χ_i is omitted, whenever its elements are all zero. Besides we put the table of $s = 1/2$ only as a reference.

a) $s = 1/2$.

$f = 2$

	χ_1	χ_2
$S = 0$	0	1
1	1	0

$f = 3$

	χ_1	χ_2
$S = 1/2$	0	1
3/2	1	0

$f = 4$

	χ_1	χ_2	χ_3
$S = 0$	0	0	1
1	0	1	0
2	1	0	0

$f = 5$

	χ_1	χ_2	χ_3
$S = 1/2$	0	0	1
3/2	0	1	0
5/2	1	0	0

$f = 6$

	χ_1	χ_2	χ_3	χ_5
$S = 0$	0	0	0	1
1	0	0	1	0
2	0	1	0	0
3	1	0	0	0

b) $s = 1$

$f = 2$

	χ_1	χ_2
$S = 0$	1	0
1	0	1
2	1	0

$f = 3$

	χ_1	χ_2	χ_3
$S = 0$	0	0	1
1	1	1	0
2	0	1	0
3	1	0	0

$f = 4$

	χ_1	χ_2	χ_3	χ_4
$S = 0$	1	0	1	0
1	0	1	0	1
2	1	1	1	0
3	0	1	0	0
4	1	0	0	0

$f = 5$

	χ_1	χ_2	χ_3	χ_4	χ_5
$S = 0$	0	0	0	1	0
1	1	1	1	0	1
2	0	1	1	1	0
3	1	1	1	0	0
4	0	1	0	0	0
5	1	0	0	0	0

$f = 6$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_8
$S = 0$	1	0	1	0	0	0	1
1	0	1	0	1	1	1	0
2	1	1	2	0	0	1	0
3	0	1	1	1	1	0	0
4	1	1	1	0	0	0	0
5	0	1	0	0	0	0	0
6	1	0	0	0	0	0	0

c) $s = 3/2$

$f = 2$

	χ_1	χ_2
$S = 0$	0	1
1	1	0
2	0	1
3	1	0

$f = 3$

	χ_1	χ_2	χ_3
$S = 1/2$	0	1	0
3/2	1	1	1
5/2	1	1	0
7/2	0	1	0
9/2	1	0	0

$f = 4$

	χ_1	χ_2	χ_3	χ_4	χ_5
$S = 0$	1	0	0	0	1
1	0	2	0	1	0
2	1	1	2	1	0
3	1	2	0	1	0
4	1	1	1	0	0
5	0	1	0	0	0
6	1	0	0	0	0

$f = 5$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6
$S = 1/2$	0	1	1	1	1	0
3/2	1	2	2	1	1	1
5/2	1	2	2	2	1	0
7/2	1	2	2	1	1	0
9/2	1	2	1	1	0	0
11/2	1	1	1	0	0	0
13/2	0	1	0	0	0	0
15/2	1	0	0	0	0	0

$f = 6$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9
$S = 0$	0	1	0	1	2	0	0	0	1
1	1	1	3	1	0	0	1	1	0
2	0	3	2	2	2	3	0	0	1
3	2	2	4	2	1	2	1	1	0
4	1	3	2	2	2	2	0	0	0
5	1	2	3	1	0	1	0	0	0
6	1	2	1	1	1	0	0	0	0
7	1	1	1	0	0	0	0	0	0
8	0	1	0	0	0	0	0	0	0
9	1	0	0	0	0	0	0	0	0

d) $s = 5/2$.

$f = 2$

	χ_1	χ_2
$S = 0$	0	1
1	1	0
2	0	1
3	1	0
4	0	1
5	1	0

$f = 3$

	χ_1	χ_2	χ_3
$S = 1/2$	0	1	0
3/2	1	1	1
5/2	1	2	1
7/2	1	2	0
9/2	1	1	1
11/2	1	1	0
13/2	0	1	0
15/2	1	0	0

$f = 4$

	χ_1	χ_2	χ_3	χ_4	χ_5
$S = 0$	1	0	2	0	1
1	0	3	0	2	0
2	2	2	3	2	1
3	1	4	1	3	0
4	2	3	3	2	1
5	1	4	1	2	0
6	2	2	2	1	0
7	1	2	0	1	0
8	1	1	1	0	0
9	0	1	0	0	0
10	1	0	0	0	0

$f = 5$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7
$S = 1/2$	1	2	2	2	2	1	0
3/2	1	4	4	4	3	2	0
5/2	2	5	6	5	4	2	1
7/2	2	6	6	6	4	2	0
9/2	3	6	6	5	4	2	0
11/2	2	6	5	5	3	1	0
13/2	2	5	5	3	2	1	0
15/2	2	4	3	3	1	0	0
17/2	2	3	2	1	1	0	0
19/2	1	2	1	1	0	0	0
21/2	1	1	1	0	0	0	0
23/2	0	1	0	0	0	0	0
25/2	1	0	0	0	0	0	0

$f = 6$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}
$S = 0$	0	2	0	3	4	2	0	0	2	0	1
1	2	3	8	4	1	7	4	3	1	1	0
2	1	8	7	8	6	11	3	1	5	1	0
3	4	7	13	9	4	12	6	5	3	1	0
4	2	10	11	11	8	13	4	2	5	1	0
5	4	9	14	9	4	13	5	3	2	1	0
6	3	10	10	10	7	10	3	2	3	0	0
7	4	8	11	7	3	8	3	2	1	0	0
8	2	8	7	6	4	6	1	0	0	0	0
9	3	5	7	4	2	3	1	1	0	0	0
10	2	5	3	3	2	2	0	0	0	0	0
11	2	3	3	1	0	1	0	0	0	0	0
12	1	2	1	1	1	0	0	0	0	0	0
13	1	1	1	0	0	0	0	0	0	0	0
14	0	1	0	0	0	0	0	0	0	0	0
15	1	0	0	0	0	0	0	0	0	0	0

3. Transformation of nuclear spin functions

As an example of applications of the above results, we can obtain the transformation of nuclear spin functions of a system including several identical (relatively) fixed nuclei with a certain spatial symmetry, each nucleous having a higher spin. The symmetry group $\mathfrak{G}$ expressing the spatial symmetry of nuclear architecture of a molecular system can be always represented with homomorphic correspondence by a permutation group $\mathfrak{P}$, which is generally a sub-group of a symmetric group $\mathfrak{S}_f$. It is easily proved, however, that the followings correspond isomorphically to each other:

$$\left.\begin{aligned}\mathfrak{P} \simeq \mathfrak{G} \equiv\ & C_i, C_s, C_2, C_{3v}, D_3, C_4, C_{4v}, V_d, D_4, D_6, \\ & T, T_d, O, O_h, \text{ etc.}\end{aligned}\right\} \tag{10}$$

Therefore, the irreducible representations of the grovp $\mathfrak{S}_f$ are not always irreducible for the representations of $\mathfrak{P}$, or isomorphically represented group $\mathfrak{G}$, and so by considering the spatial symmetry of equivalent nuclear system the irreducible representations of $\mathfrak{S}_f$ can be reduced in general to those of $\mathfrak{G}$. If $\mathfrak{P}$ is homomorphic to $\mathfrak{G}$, the reduction must be carried out in regard to the class elements corresponding to the permutations of equivalent nuclei. And in the case of $\mathfrak{P} \equiv \mathfrak{S}_f$, the reduction is not necessary. The correspondence of the irreducible representations of $\mathfrak{S}_f$ to the typical $\mathfrak{G}$ is given by the following Table II.

TABLE II.

a) $\mathfrak{S}_2$.

$\mathfrak{G} \equiv C_i$	C_s	C_2	C_{2v}	D_{2v}	$D_{\infty h}$
χ_1	A_g	A'	A	A_1	A_{1g}	Σ_g^+
χ_2	A_u	A''	B	B_2	B_{2u}	Σ_u^+

b) $\mathfrak{S}_3$.

$\mathfrak{G} \equiv C_{3v}, D_3$	D_3
χ_1	A_1	A_{1g}
χ_2	E	E_g
χ_3	A_2	A_{2g}

c) $\mathfrak{S}_4$.

$\mathfrak{G} \equiv C_{4v}, D_4, V_d$	D_{4h}	T_d
χ_1	A_1	A_{1g}	A_1
χ_2	$B_1 + E$	$B_{1g} + E_{1u}$	T_2
χ_3	$A_1 + B_1$	$A_{1g} + B_{2g}$	E
χ_4	$A_2 + E$	$A_{2g} + E_{1u}$	T_1
χ_5	B_1	B_{2g}	A_2

d) $\mathfrak{S}_6$.

$\mathfrak{G} \equiv D_{6v}$:

$\chi_1 = A_{1g}$, $\chi_2 = B_{1u} + E_{1g} + E_{2u}$, $\chi_3 = 2A_{1g} + B_{2u} + 2E_{1g} + E_{2u}$,
$\chi_4 = 2A_{2g} + B_{1u} + B_{2u} + E_{1g} + 2E_{2u}$, $\chi_5 = A_{2g} + 2B_{1u} + E_{2u}$,
$\chi_6 = A_{1g} + A_{2g} + B_{1u} + B_{2u} + 3E_{1g} + 3E_{2u}$,
$\chi_7 = A_{1g} + A_{2g} + 2B_{2u} + 2E_{1g} + E_{2u}$,
$\chi_8 = 2A_{1g} + E_{1g} + B_{2u}$, $\chi_9 = A_{2g} + 2B_{1u} + E_{1g} + 2E_{2u}$,
$\chi_{10} = A_{1g} + E_{1g} + E_{2u}$, $\chi_{11} = B_{1u}$.

$\mathfrak{G} \equiv O_h$:

$\chi_1 = A_{1g}$, $\chi_2 = E_g + T_{1u}$, $\chi_3 = A_{1g} + E_g + T_{2g} + T_{2u}$,
$\chi_4 = A_{2g} + T_{1g} + T_{1u} + T_{2u}$, $\chi_5 = A_{2g} + A_{2u} + T_{1u}$,
$\chi_6 = E_g + E_u + T_{1g} + T_{1u} + T_{2g} + T_{2u}$, $\chi_7 = A_{1u} + T_{1g} + T_{2g} + T_{2u}$,
$\chi_8 = A_{1g} + A_{1u} + T_{2g}$, $\chi_9 = A_{2u} + E_u + T_{1u} + T_{1g}$,
$\chi_{10} = E_u + T_{2g}$, $\chi_{11} = A_{2u}$.

By making use of the above relations, the transformation of nuclear spin functions is given in terms of a direct sum of irreducible representations Γ_j of $\mathfrak{G}$: the results are generally obtained by the formula:

$$\sum_S (2S + 1) \sum_j \gamma(j, S)\, \Gamma_j , \tag{11}$$

where the factor $(2S + 1)$ is the nuclear spin multiplicity corresponding to a given resultant spin S, and $\gamma(j, S)$ are always positive integer, or the number of times in which each irreducible representation of $\mathfrak{G}$ specified by j appears.

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